

CP violation & Axions

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Outline of Lectures

- Anomalies, instantons: 2D warm up
- QCD instantons & $U(1)_A$ puzzle
- Solutions to the strong CP problem
- Invisible axion
- Axion Cosmology
- Astrophysical axion bounds
- Searches for axion

Suggested Literature

- M. Dine, "TASI lectures on the strong CP problem", hep-ph/0011376
- R.D. Peccei, "The strong CP problem & axions", hep-ph/0607268
- P. Sikivie, "Axion cosmology", astro-ph/0610440
- G.G. Raffelt, "Astrophysical axion bounds", hep-ph/0611350
- R. Battesti et al., "Axion searches in the past, at present, and in the near future"
arXiv: 0705.0615

Counting CP-violating phases in the Standard Model

4 complex 3×3 mass matrices for up, down-quarks and ℓ, ν -leptons (in the interaction basis)

$$\left. \begin{aligned} \hat{M}^{(u)}_{ij} \bar{U}'_{Li} U'_{Rj} + h.c. \\ \hat{M}^{(d)}_{ij} \bar{d}'_{Li} d'_{Rj} + h.c. \\ \hat{M}^{(\ell)}_{ij} \bar{\ell}'_{Li} \ell'_{Rj} + h.c. \end{aligned} \right\}$$

fermion-Higgs Yukawa interactions

Dirac masses

$$\frac{1}{2} M^{(\nu)}_{ij} \bar{\nu}'_{Li} \nu'^c_{Lj} + h.c. \Leftrightarrow \text{e.g. Higgs Weinberg operator}$$

Majorana masses

Here, $U'_{Li,R} = (u', c', t')_{L,R}$, $d'_{Li,R} = (d', s', b')_{L,R}$, $\ell'_{Li,R} = (e', \mu', \tau')_{L,R}$

$$\nu'_{Li} = (\nu'_e, \nu'_\mu, \nu'_\tau)_L$$

Naïve counting: $4 \times (3 \times 3) = 36$ real + $4 \times (3 \times 3) = 36$ complex

Not all physical!

Counting CP-violating phases in the Standard Model

Change basis to mass eigenstate basis via (chiral) unitary transformations:

$$U_{Li} = \hat{V}_{Lij}^{(u)} U'_{Lj}, \quad U_{Ri} = \hat{V}_{Rij}^{(u)} U'_{Rj},$$

$$d_{Li} = \hat{V}_{Lij}^{(d)} d'_{Lj}, \quad d_{Ri} = \hat{V}_{Rij}^{(d)} d'_{Rj},$$

$$e_{Li} = \hat{V}_{Lij}^{(e)} e'_{Lj}, \quad e_{Ri} = \hat{V}_{Rij}^{(e)} e'_{Rj},$$

$$\nu_{Li} = \hat{V}_{Lij}^{(\nu)} \nu'_{Lj}$$

7 3×3 unitary matrices

$$\hat{V}_{L,R}^{(F)} \cdot \hat{V}_{L,R}^{(F)\dagger} = \mathbb{1}$$

Each contains:

$$\frac{n(n-1)}{2} = |n=3| = 3 \text{ real}$$

$$+ \frac{n(n+1)}{2} = |n=3| = 6 \text{ complex parameters}$$

Counting CP-violating phases in the Standard Model

In the mass eigenstate basis we are left with:

Masses $\Rightarrow 12 = 3 + 3 + 3 + 3$ real parameters
up, down, e, ν

$$V_L^{(u)} \cdot \hat{M}^{(u)} \cdot V_R^{(u)\dagger} = \text{diag}(m_u, m_c, m_t) \in \mathbb{R}_+$$

$$V_L^{(d)} \cdot \hat{M}^{(d)} \cdot V_R^{(d)\dagger} = \text{diag}(m_d, m_s, m_b) \in \mathbb{R}_+$$

$$V_L^{(e)} \cdot \hat{M}^{(e)} \cdot V_R^{(e)\dagger} = \text{diag}(m_e, m_\mu, m_\tau) \in \mathbb{R}_+$$

$$V_L^{(\nu)} \cdot \hat{M}^{(\nu)} \cdot V_L^{(\nu)\dagger} = \text{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}) \in \mathbb{R}_+$$

Charged current interactions:

$$\text{quarks: } \propto \bar{U}_L; \gamma^\mu V_{CKMij} d_L; W_\mu^+, \quad V_{CKM} = V_L^{(u)} V_L^{(d)\dagger}$$

$$\text{leptons: } \propto \bar{\nu}_L; \gamma^\mu V_{PMNSij} e_L; W_\mu^+, \quad V_{PMNS} = V_L^{(\nu)} V_L^{(e)\dagger}$$

Counting CP-violating phases in the Standard Model

$$\hat{V}_{\text{CKM}} \cdot \hat{V}_{\text{CKM}}^{\dagger} = \mathbb{1} \quad (3 \text{ real} + 6 \text{ complex})$$

$$\hat{V}_{\text{PMNS}} \cdot \hat{V}_{\text{PMNS}}^{\dagger} = \mathbb{1} \quad (3 \text{ real} + 6 \text{ complex})$$

$\hat{V}_{\text{CKM}}$: 5 complex parameters can be removed by redefinition of U_{Li}, d_{Li} & U_{Ri}, d_{Ri} phases, so that masses remain real $\Rightarrow$ 1 physical phase

$\hat{V}_{\text{PMNS}}$: 3 complex parameters can be removed by redefinition of ℓ_{Li}, ℓ_{Ri} phases. Redefinition of ν_{Li} phases would make Majorana masses complex $\Rightarrow$ 3 physical phases

Hence, 1 (quark sector) + 3 (lepton sector) = 4 CP-violating phases!

Counting CP-violating phases in the Standard Model

The above counting is correct at 'classical' level

Quantum corrections provide an obstruction to some chiral transformations (change basis), known as quantum anomaly $\Rightarrow$

There is one extra CP-violating parameter in the Standard Model

Chiral anomaly: QCD with one quark

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \underbrace{i \bar{q}'_L \gamma^\mu D_\mu q'_L + i \bar{q}'_R \gamma^\mu D_\mu q'_R}_{i \bar{q}' \gamma^\mu D_\mu q'}$$

$$\underbrace{- m \bar{q}'_L q'_R - m^* \bar{q}'_R q'_L}_{- M \cos \theta \bar{q}' q' - M \sin \theta \bar{q}' \gamma_5 q'}$$

$$q'_{L,R} = \frac{1 \mp \gamma_5}{2} q', \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_\mu = \partial_\mu + i g G_\mu^a T^a$$

$$q' = q'_L + q'_R$$

$$m = M e^{i\theta}$$

physical mass of q -quark

• Is θ a physical parameter?

Chiral anomaly: QCD with one quark

• Let's try to remove θ :

$$q = e^{\frac{i}{2}\gamma_5} q' \Rightarrow \begin{cases} q_L = e^{-i\frac{\theta}{2}} q'_L \\ q_R = e^{+i\frac{\theta}{2}} q'_R \end{cases}$$

$$\mathcal{L}_{\text{QCD}}(G_\mu^a, q', \theta) = \mathcal{L}_{\text{QCD}}(G_\mu^a, q, \theta=0)$$

Exercise: Verify this explicitly

Complex phase θ is removed from the classical Lagrangian!

Chiral anomaly: QCD with one quark

In quantum theory the relevant object is the generating functional:

$$Z = \int D G_{\mu}^a \underbrace{D q' D \bar{q}'}_{\text{QCD}} e^{i \underbrace{\int d^4 x \mathcal{L}_{\text{QCD}}}_{\text{QCD}}}$$

- Chiral phase transformation $q = e^{\frac{i}{2} \theta \gamma_5} q'$ is not a canonical transformation

$$D q' D \bar{q}' = J(\theta) D q D \bar{q}$$

θ -dependent Jacobian

$$J(\theta) = \exp \left\{ i \int d^4 x \frac{\theta g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} \right\}$$

Chiral anomaly: QCD with one quark

$$\mathcal{L}_{\text{QCD}}(G_r^a, q', \theta) = \mathcal{L}_{\text{QCD}}(G_r^a, q, \theta=0) + \frac{\theta g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$$

where $\tilde{G}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^a$ (dual field strength)

Note: $G_{\mu\nu}^a \tilde{G}^{a\mu\nu} = \partial_\mu K^\mu$

$$K^\mu = \epsilon^{\mu\nu\rho\sigma} G_\nu^a \left[G_{\rho\sigma}^a - \frac{g}{3} f^{abc} G_\rho^b G_\sigma^c \right]$$

$$\int d^4x \partial_\mu K^\mu = \int d\tau K^\mu \neq 0! \quad (\text{instantons, topologically non-trivial vacuum})$$

Thus, the complex phase θ can not be removed in quantum theory! $\Leftrightarrow$ Extra CP-violating parameter

Analytic QCD - the CP^N model in 2D

(Peskin & Schroeder, An Intro to QFT, prob. 13.3)

Consider a set of complex scalar fields z_i , $i=1, 2, \dots, N+1$, which satisfy the following constraints:

$$(1) \sum_i |z_i|^2 = 1$$

$$(2) z_i \sim e^{i\alpha} z_i$$

$\left\{ \begin{array}{l} \text{CP}^N \text{ space} \end{array} \right.$

(1) Implemented by adding a Lagrange multiplier field, $\lambda(x)$

(2) The identification $z_i \sim e^{i\alpha} z_i$ must be valid at every space-time point $\Rightarrow$ $U(1)$ gauge invariance

Analogue QCD - the CP^N model in 2D

Hence,
$$\mathcal{L}_{CP^N} = \frac{1}{g^2} \left[\sum_{i=1, \dots, N} |D_\mu z_i|^2 - \lambda(x) (|z_i|^2 - 1) \right]$$

$$D_\mu = \partial_\mu + i A_\mu$$

Note: no kinetic terms for A_μ & λ . We can solve the respective algebraic equations of motion:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \Rightarrow A_\mu = i \frac{z_i^* \partial_\mu z_i - z_i \partial_\mu z_i^*}{2 |z_i|^2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow |z_i|^2 = 1$$

Thus we have an equivalent Lagrangian:
$$\mathcal{L}'_{CP^N} = \frac{1}{g^2} \left[|\partial_\mu z_i|^2 - 2 |z_i^* \partial_\mu z_i|^2 \right]$$

Analogue QCD - the CP^N model in 2D

- Like pure/massless QCD, the CP^N model in 2D is classically scale invariant - there are no dimensionfull parameters $[z_i] = m^0$, $[\lambda] = m^2$, $[A_\mu] = m^1$

- Let's integrate out z_i fields in the functional integral:

$$Z = \int Dz_i Dz_i^* DA_\mu D\lambda e^{\frac{i}{g^2} \int d^2x (|D_\mu z_i|^2 - \lambda(|z_i|^2 - 1))}$$

$$= \int Dz_i Dz_i^* DA_\mu D\lambda e^{-\int d^2x z_i^* \frac{i(D_\mu D^\mu + \lambda)}{g^2} z_i + \int d^2x \frac{i\lambda}{g^2}}$$

$$Dz = \frac{dz(x_1)}{\sqrt{\pi}} \frac{dz(x_2)}{\sqrt{\pi}} \dots$$

Gaussian integral

$$\int \frac{dz dz^*}{\pi} e^{-z^* A z} = \frac{1}{A}$$

$$= \int DA_\mu D\lambda \det \left[\frac{i}{g^2} (D_\mu D^\mu + \lambda) \right]^{-(N+1)} e^{+\int d^2x \frac{i\lambda}{g^2}}$$

$$= \left| \text{Use } \det \hat{A} = e^{\text{Tr} \log \hat{A}} \right|$$

Analogue QCD - the CP^N model in 2D

$$Z \propto \int DA_\mu D\lambda e^{i S_{\text{eff}}}$$

$$S_{\text{eff}} = i N \text{Tr} \log (D_\mu D^\mu + \lambda) + \frac{1}{g^2} \int d^2x \lambda$$

Note: $N+1 \approx N$ (in large N limit)

$$\text{Tr}(\dots) = \langle x | \dots | x \rangle$$

- The rest of the integral can be solved 'exactly' in the 't Hooft large N limit: $N \rightarrow \infty$, $g^2 \rightarrow 0$, $Ng^2 = \text{const.}$

Saddle point approximation is exact in the large N limit

Analogue QCD - the CP^N model in 2D

$$(1) \quad 0 = \frac{\delta S_{\text{eff}}}{\delta A_\mu} = -N \text{Tr} \frac{2D_\mu}{D_\mu D^\mu + \lambda} = -N \langle x | \frac{2(\partial_\mu + iA_\mu)}{(\partial_\mu + iA_\mu)^2 + \lambda} | x \rangle$$

$$(2) \quad 0 = \frac{\delta S_{\text{eff}}}{\delta \lambda} = -N \text{Tr} \frac{1}{D_\mu D^\mu + \lambda} + \frac{i}{g^2}$$

(1) $\Rightarrow A_\mu = 0$ Exercise: Demonstrate this explicitly

$$(2) \quad \langle x | \frac{1}{\partial_\mu \partial^\mu + \lambda} | x \rangle = + \frac{i}{g^2 N}$$

$$\int \frac{d^2 k}{(2\pi)^2} \langle x | \frac{1}{\partial_\mu \partial^\mu + \lambda} | k \rangle \langle k | x \rangle = + \frac{i}{g^2 N}$$

$$\boxed{\int \frac{d^2 k}{(2\pi)^2} \frac{1}{k_\mu k^\mu - \lambda} = - \frac{i}{g^2 N}}$$

Analogue QCD - the CP^N model in 2D

Integral is divergent, - analytic continuation $k_0 \rightarrow i k_0$ & cut-off regularization

$$\frac{1}{(2\pi)^2} \int_0^{M\sqrt{N}} \int_0^{2\pi} \frac{dk_E^2 d\theta}{2} \frac{1}{k_E^2 + \lambda} = \frac{1}{g^2 N}$$

$$\Downarrow$$

$\lambda \simeq M^2 e^{-\frac{4\pi}{g^2 N}}$

Dimensional transmutation - emergence of a mass scale in classically scale invariant theory due to the quantum scale anomaly $\Leftrightarrow$ similar to QCD!

Analogue QCD - the CP^N model in 2D

A kinetic term for A_μ is generated at $\sim O(1/N)$ level:

$\frac{1}{k^2 - \lambda_0}$, $\lambda_0 \equiv m^2 = M^2 e^{-\frac{4\pi}{3N}}$

$$\mathcal{L}_A = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$e^2 = \frac{3\pi M^2}{N}$$

- Also add light ($m_q < m$) 'quark' & CP-violating term:

$$\mathcal{L}_{AQCD} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \underbrace{\frac{\theta}{2\pi} \int d^2x \epsilon_{\mu\nu} F^{\mu\nu}}_{CP} + i\bar{q} \gamma^\mu D_\mu q - m_q \bar{q} q$$

Analogue QCD - the CP^N model in 2D

Various important implications follow from this simple model:

① Chiral anomaly

Vector current: $j^\mu = \bar{q}(x) \gamma^\mu q(x)$, $\partial_\mu j^\mu = 0$ (classical + quantum $\equiv$ gauge invariance)

Chiral current: $j_5^\mu = \bar{q}(x) \gamma^\mu \gamma^5 q(x)$; $\partial_\mu j_5^\mu \big|_{m_q \rightarrow 0} = 0$ (classical)

$\partial_\mu j_5^\mu \big|_{m_q \rightarrow 0} \neq 0$ (quantum)

$\hat{A}(x) \hat{B}(x)$ ill-defined in QFT
needs regularization!

Analogue QCD - the CP^N model in 2D

- Employ point-splitting regularization

$$j_5^\mu(x) = \bar{q}(x+\epsilon) \underbrace{e^{i \int_x^{x+\epsilon} A_\mu dx^\mu}}_{\text{Wilson line to restore gauge invariance}} \gamma^\mu \gamma^5 q(x)$$

Exercise: Show that this operator is gauge invariant

$$= \bar{q}(x+\epsilon) \gamma^\mu \gamma^5 q(x) + i \epsilon_\mu A^\mu \bar{q}(x+\epsilon) \gamma^\mu \gamma^5 q(x) + O(\epsilon^2)$$

- Compute vacuum expectation value, using the propagator for massless fermion:

$$\langle 0 | \bar{q}(x+\epsilon) q(x) | 0 \rangle \Big|_{m_f \rightarrow 0} = \frac{1}{2\pi} \frac{\epsilon_\alpha \gamma^\alpha}{\epsilon^2 = (\epsilon_P \epsilon^P)}$$

Analogue QCD - the CP^N model in 2D

$$\langle 0 | j_5^\mu | 0 \rangle = \frac{1}{2\pi} \frac{\epsilon_\alpha}{\epsilon^2} \text{Tr}(\gamma^\alpha \gamma^\mu \gamma^5) + \frac{i}{2\pi} \epsilon_\rho \epsilon_\alpha \frac{1}{\epsilon^2} A^\rho \text{Tr}(\gamma^\alpha \gamma^\mu \gamma^5)$$

$$= \left\{ \begin{array}{l} i\gamma^\mu \gamma^5 = \epsilon^{\mu\nu} \gamma_\nu \text{ (Exercise: Verify this)} \\ \gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu_\nu \mathbb{1} \\ \text{Tr} \gamma^\alpha \gamma_\mu = \text{Tr} \gamma_\nu \gamma^\alpha = 2\delta^\alpha_\nu \\ \text{Tr}(\gamma^\alpha \gamma^\mu \gamma^5) = -i\epsilon^{\mu\nu} \text{Tr}(\gamma^\alpha \gamma_\nu) = -2i\epsilon^{\mu\alpha} \end{array} \right.$$

$$= \frac{1}{2\pi} \frac{\epsilon^{\mu\alpha} \epsilon_\alpha}{\epsilon^2} + \frac{\epsilon^{\mu\alpha}}{\pi} \frac{\epsilon_\alpha \epsilon_\rho}{\epsilon^2} A^\rho = \frac{1}{2} \eta_{\alpha\rho} \epsilon^2$$

$$= \frac{1}{2\pi} \frac{\epsilon^{\mu\alpha} \epsilon_\alpha}{\epsilon^2} + \frac{1}{2\pi} \epsilon^{\mu\alpha} A_\alpha$$

$$\langle 0 | \partial_\mu j_5^\mu | 0 \rangle = \frac{1}{2\pi} \epsilon^{\mu\alpha} \partial_\mu A_\alpha = \frac{1}{4\pi} \epsilon^{\mu\alpha} F_{\mu\alpha}$$

Analogous QCD - the CP^N model in 2D

- Thus, anomalous non-conservation of j_5^μ is proportional to the ~~CP~~ θ -term operator, $\frac{\theta}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}$, in the effective Lagrangian
- Next, let's inspect the θ -term:

$$\int d^2x \epsilon^{\mu\nu} F_{\mu\nu} = 2 \int d^2x \partial_\mu (\epsilon^{\mu\nu} A_\nu) = 2 \underbrace{\int d^2x \epsilon^{\mu\nu} A_\nu}_{\text{boundary}}$$

Are there field configurations A_μ for which the above θ -term is non-vanishing?

Analogue QCD - the CP^N model in 2D

- Consider 'vacuum' Maxwell's equation

$$\partial_\mu F^{\mu\nu} = 0$$

Fix the gauge: $A_\mu(x) = 0$

$$\partial_1^2 A_0 = 0 \quad \& \quad \partial_0 \partial_1 A_0 = 0$$

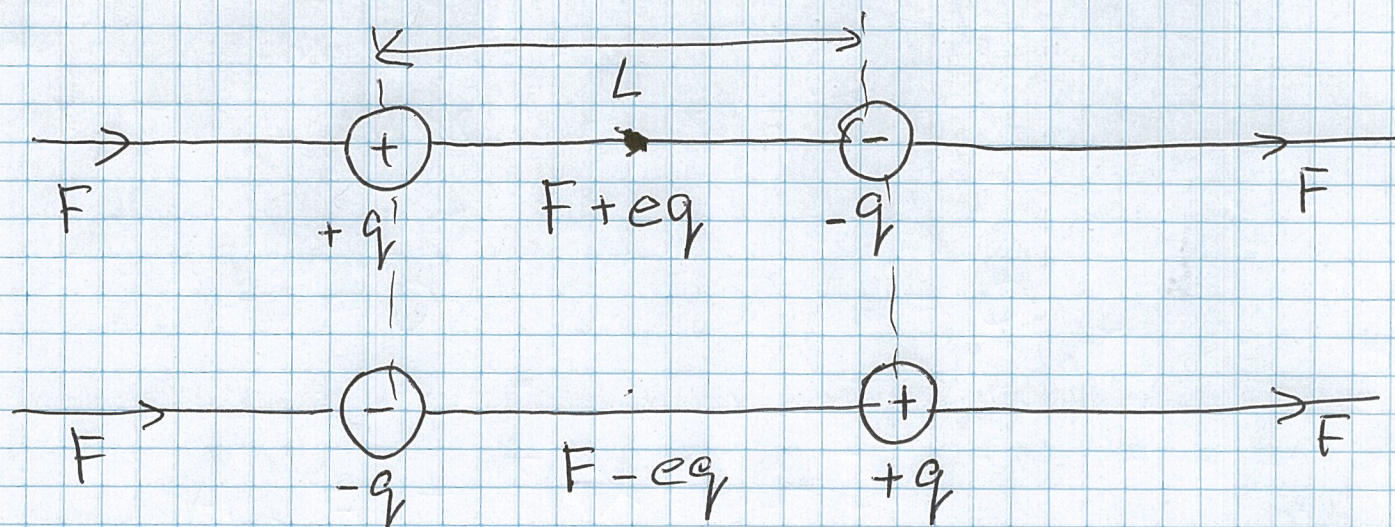
$$A_0 = -F \cdot \underset{(x')}{x} + G \quad \swarrow \text{irrelevant constant}$$

$$F_{01} = -\partial_1 A_0 = F \equiv \text{const}$$

Hence, constant background electric field supports θ -term!

Analogue QCD - the CP^N model in 2D

- The configuration with a constant electric field F in 2D may be the ground (minimal energy) state:



- Compute energy difference:

$$\Delta E = \frac{1}{2} \int d^2x (F \pm eq)^2 - F^2 = \frac{1}{2} L [(F+eq)^2 - F^2]$$

For $|F| > \frac{1}{2}eq$, $\Delta E > 0 \Rightarrow$ Background field F without pair production is energetically favourable!

Analogue QCD - the CP^N model in 2D

- Thus A can be identified with F :

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + e \epsilon_{\mu\nu} F$$

$$\mathcal{L}_{\text{AQCD}} = -\frac{1}{4e^2} (F_{\mu\nu} + e \epsilon_{\mu\nu} F)^2 + \dots = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{F}{2e} \epsilon_{\mu\nu} F^{\mu\nu} + \dots$$

$$\boxed{F = \frac{e\theta}{\pi}}$$

(Coleman 7b')

Note! physics is periodic in θ , $\theta \sim \theta + 2\pi$

Analogue QCD - the CP^N model in 2D

- Next, we consider instantons - Euclidean field configurations, which describe quantum tunneling between topologically distinct vacua.
- Analytic continuation to Euclidean space: $x_0 \rightarrow ix_0$

$$iS \rightarrow i(i)S_E = -S_E$$

$$\mathcal{L}'_{CP^N} = \frac{1}{g^2} \left[|\partial_\mu z_i|^2 - |z_i^* \partial_\mu z_i|^2 \right] \rightarrow$$

$$\rightarrow \boxed{\mathcal{L}_{CP^N}^{IE} = + \frac{1}{g^2} \left[|\partial_\mu z_i|^2 + |z_i^* \partial_\mu z_i|^2 \right]}$$

Analogue QCD - the CP^N model in 2D

- Consider $N=1$ case (CP^1 model)

2 complex scalar fields, z_1, z_2 with:

$$|z_1|^2 + |z_2|^2 = 1, \quad z_i \sim e^{i\alpha} z_i$$

$\Downarrow$
Only 2 real independent fields, $CP^1 \sim S^2$

- Change parameterisation:
$$n^a = z_i^* (\sigma^a)_{ij} z_j, \quad a=1,2,3$$
$$n^a n^a = 1 \quad \text{--- Pauli matrices}$$

The Lagrangian $\mathcal{L}_{CP^1}^{I/E}$ becomes (Exercise: Verify this)

$$\mathcal{L}_{NLO(3)} = \frac{1}{g^2} (\partial_\mu n^a) (\partial^\mu n^a), \quad n^a n^a = 1$$

Analogue QCD - the CP^N model in 2D

• Topology (Belavin & Polyakov 75')

Finite action solutions $\Rightarrow n^a \rightarrow (1, 0, 0)$ as $r \rightarrow \infty$

Hence, they represent a mapping:

$$n^a : \mathbb{R}^2 \cup \{\infty\} \sim S_x^2 \rightarrow S_{n^a}^2$$

- All such solutions are characterized by ~~a~~ a degree of mapping (winding number) $Q \in \mathbb{Z}_+$ = how many times $S_{n^a}^2$ is wrapped by S_x^2 through the map

$$Q = \frac{1}{8\pi} \int d^2x \epsilon_{\mu\nu} \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c \in \mathbb{Z}_+$$

This is a topological charge $\equiv$ invariant under small deformations of n^a .

Analogue QCD - The CP^N model in 2D

Physical significance: Since instantons describe quantum transition from vacuum-to-vacuum with change in Q , there exist topologically distinct vacua (with different Q) and the physical vacuum is represented as a superposition of thereof.

Analogue QCD - the CP^N model in 2D

- Instanton solutions; Bogomolnyi trick:

$$\left(\partial_\mu h^a \pm \epsilon_{\mu\nu} \epsilon_{abc} h^b \partial^\nu h^c \right)^2 \geq 0$$

$\Downarrow$

$$\frac{1}{g^2} (\partial_\mu h^a) (\partial^\mu h^a) \geq \left| \frac{1}{g^2} \epsilon_{\mu\nu} \epsilon_{abc} h^a \partial^\mu h^b \partial^\nu h^c \right|$$

$\Downarrow$

$$S \geq \frac{8\pi}{g^2} |Q|$$

Saturated when:

$$\partial_\mu h^a = \mp \epsilon_{\mu\nu} \epsilon_{abc} h^b \partial^\nu h^c$$

Selfduality
equation

Analogue QCD - the CP^N model in 2D

Explicit solution for $Q=1$ instanton
with boundary condition $h^a = (1, 0, 0) |_{r \rightarrow \infty}$:

$$\frac{h^2 + i h^3}{1 - h^1} = \frac{r e^{i\theta}}{\lambda}$$

(r, θ) polar coordinates of 2D
space

$$h^{12} + h^{22} + h^{32} = 1$$

λ is a size of instanton
not fixed in the semiclassical approximation
due to the scale invariance

Exercise: Prove that $Q=1$ for the above configuration

Analogue QCD - the CP^N model in 2D

- Given classical instanton solution breaks symmetries of space-time translations, rotations, scale transformations & special conformal invariance.

Typical quantum computations involve perturbations of the classical solutions (zero modes of broken symmetry) and one has to integrate over those perturbations in the functional integral. E.g., integration over the size parameter λ yields a factor ('t Hooft 76)

$$\propto \int d\lambda \lambda^{-1} e^{-\frac{8\pi}{g^2(\lambda)}}$$

- Asymptotically free theory $\Rightarrow g(\lambda) \xrightarrow{\lambda \rightarrow 0} 0 \Rightarrow$ the dominant contribution comes from 'large' instantons

Real QCD - the strong CP problem

Recap of CP^N model;

- Physical CP violating term $\propto F_{\mu\nu} \tilde{F}^{\mu\nu}$ is allowed in the effective Lagrangian
- This term can be interpreted as being due to a constant background electric field. For certain values of this constant field is a ground state of the system.
- There are multiple vacuum states, labelled by integer (topological) numbers. Quantum transition between these states are generated by instantons. Hence, the true vacuum state is a superposition of multiple vacua - the so-called θ -vacuum. CP-violating term is supported by this vacuum structure

Real QCD - The Strong CP problem

Several features of CP^N model is shared by QCD:

- Multiple vacuum states identified by a topological charge

$$n = \int d^3x K^0$$

$$K^\mu = \epsilon^{\mu\nu\sigma\rho} G_{\nu\sigma} \left[G_{\rho\sigma} - \frac{g}{3} f_{abc} G_{\sigma\rho} G_{ab} \right]$$

gluon fields gluon field strengths Instanton

- Instanton configurations provide quantum tunneling between topologically distinct n -vacua:

$$n \Big|_{t \rightarrow +\infty} - n \Big|_{t \rightarrow -\infty} = \frac{g^2}{32\pi^2} \int d\tau K^\mu \Big|_{t=-\infty}^{t=+\infty} \neq 0$$

Real QCD - the strong CP problem

• Thus the true vacuum is:

$$|\theta\rangle = \sum_n e^{-in\theta} |n\rangle$$

$$\langle\theta|\theta\rangle_{t=+\infty} = \sum_{m,n} e^{im\theta} e^{-in\theta} \langle m|n\rangle =$$

$$= \sum_{m,n} e^{i\nu\theta} \langle n+\nu|n\rangle$$

$$\nu = \frac{g^2}{32\pi^2} \int d\tau_\mu K^\mu \Big|_{t=-\infty}^{t=+\infty} = \frac{g^2}{32} \int d^4x G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$$

Real QCD - the strong CP problem

Recall functional integral representation of the vacuum-to-vacuum transition amplitude:

$$\begin{aligned} \langle \Theta | \Theta \rangle_{t=+\infty, t=-\infty} &= \int \mathcal{D}G_\mu^a \mathcal{D}q' \mathcal{D}\bar{q}' e^{iS_{\text{QCD}}(G_\mu^a, q', \bar{q}')} \delta\left(v - \frac{g^2}{32\pi^2} \int d^4x G_\mu^a \tilde{G}^{a\mu\nu}\right) \\ &= \int \mathcal{D}G_\mu^a \mathcal{D}q' \mathcal{D}\bar{q}' e^{iS_{\text{QCD}} + \frac{i\Theta g^2}{32\pi^2} \int d^4x G_\mu^a \tilde{G}^{a\mu\nu}} \end{aligned}$$

In addition, we have learned that to remove complex phases from quark ~~sector~~^{masses} we need to perform chiral rotations, which result in another $\propto G_\mu^a \tilde{G}^{a\mu\nu}$ term due to the chiral anomaly:

$$\boxed{\mathcal{L}_{\text{eff QCD}} = \mathcal{L}_{\text{QCD}}(G_\mu^a, q, \bar{q}) + \frac{\bar{\Theta} g^2}{32\pi^2} G_\mu^a \tilde{G}^{a\mu\nu}}$$

Real QCD - the strong CP problem

- Thus the physical CP-violating $\bar{\theta}$ -term is given by:

$$\bar{\theta} = \theta + \text{Arg det}(\hat{M}_u \hat{M}_d)$$

Before discussing the problem with instantons in QCD, let's discuss good things they do for us.

Real QCD - the strong CP problem

- Consider QCD with $N_f = 2$ flavours of quarks only - good low energy approximation since $m_{u,d} \ll \Lambda_{\text{QCD}}$

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \bar{q} = (\bar{u}, \bar{d})$$

- In the limit $m_{u,d} \rightarrow 0$, classical global chiral symmetry holds:

$$\underbrace{SU(2)_L \times SU(2)_R \times \underbrace{U(1)_{L+R}}_{U(1)_B}}_{\text{exact quantum mechanically}} \times \underbrace{U(1)_{L-R}}_{U(1)_A}_{\text{anomalous}}$$

Real QCD - the strong CP problem

Quark condensate:

$$\langle 0 | (\bar{q}_L^a q_R^b + \bar{q}_R^a q_L^b) | 0 \rangle = F \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{flavour space} \\ a, b = u, d}}$$

Spontaneous global symmetry breaking

$$SU(2)_L \times SU(2)_R \times U(1)_{\cancel{B}} \times U(1)_A \rightarrow \\ \rightarrow SU(2)_{L+R} \times U(1)_B$$

Compute # of Goldstone bosons:

$$\underbrace{(3 + 3 + 1 + 1)}_{\substack{\text{\# of initial generators} \\ \text{of chiral symmetry}}} - \underbrace{(3 + 1)}_{\substack{\text{\# of generators of} \\ \text{unbroken symmetry}}} = \underline{\underline{4}}$$

Real QCD - the strong CP problem

4 (pseudo)Goldstone fields parametrise the coset space:

$$SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A / SU(2)_{L+R} \times U(1)_B$$

Composite operator:

$$\bar{q} q = F e^{i \frac{\vec{\pi}(x) \cdot \vec{\sigma} + \eta(x) \sigma^0}{f_\pi}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3), \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$i=1,2,3$

Expected Goldstone bosons: $\pi^\pm = \pi^1 \pm i\pi^2$
 π^0, η^0

Note that the chiral symmetry is also broken explicitly by quark masses $\hat{M}_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \Rightarrow \pi^0, \eta$ must pick up small masses

Real QCD - the strong CP problem

Inspect quark mass term in the QCD Lagrangian:

$$\begin{aligned} \mathcal{L}_{M_q} &= \bar{q}_L^a M_q^{ab} q_R^b = F \text{Tr} \left(e^{i \frac{\vec{\pi} \cdot \vec{\sigma} + \eta 5^0}{f_\pi}} \cdot \hat{M}_q \right) \\ &= F \text{Tr} \left[\left(1 + \frac{i \vec{\pi} \cdot \vec{\sigma}}{f_\pi} - \frac{\vec{\pi} \cdot \vec{\pi} \sigma_i \sigma_i}{2 f_\pi^2} + \dots \right) \left(1 + \frac{i \eta 5^0}{f_\pi} - \frac{\eta^2 5^0}{2 f_\pi^2} + \dots \right) \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \right] \\ &= - \frac{F(m_u + m_d)}{f_\pi^2} (\vec{\pi} \cdot \vec{\pi} + \eta^2) + \dots \end{aligned}$$

Hence, $M_{\vec{\pi}, \eta}^2 = - \frac{(m_u + m_d) F}{f_\pi^2} \ll - \frac{F}{f_\pi}$, since $\frac{m_u + m_d}{f_\pi} \ll 1$

Observations: $M_{\vec{\pi}}^2 \ll M_\eta^2$

$$F \approx (-250)^3 \text{ MeV}, \quad f_\pi \approx 100 \text{ MeV}$$

Real QCD - the strong CP problem

η -meson is not a (pseudo) Goldstone boson,

$$m_\eta \Big|_{m_u, m_d \rightarrow 0} \neq 0!$$

This was known as the missing Goldstone or $U(1)_A$ problem of QCD

The resolution of the $U(1)_A$ puzzle comes from instantons ('t Hooft 76) - instanton-mediated transitions violate $U(1)_A$ explicitly through the anomaly & provide additional source for η -meson mass.

Real QCD - the strong CP problem

Instantons in QCD can be found much as in the CP^N model

Euclidean action is bounded:

$$\int (G_{\mu\nu}^a \pm \tilde{G}_{\mu\nu}^a)^2 d^4x = \int (\underbrace{G_{\mu\nu}^a G^{a\mu\nu}}_{= \underbrace{\tilde{G}_{\mu\nu}^a \tilde{G}^{a\mu\nu}}_{= 2G_{\mu\nu}^a G^{a\mu\nu}}}) \geq 0$$

$$\Downarrow$$
$$\int (G_{\mu\nu}^a \pm \tilde{G}_{\mu\nu}^a) \geq 0$$

Hence, action is minimal if:

$$G_{\mu\nu}^a = \pm \tilde{G}_{\mu\nu}^a \quad (\text{selfduality equation})$$

Real QCD - the strong CP problem

Take $SU(2)$ gauge theory for definiteness (can be embedded into $SU(3)_c$)

Solution:

$$\cancel{G}^a_r \sigma^a = f(r^2) g \partial_r g^{-1}$$

Belavin, Polyakov, Schwartz
Tyupkin (BPST) 75'

$$g = \frac{X_4 + i \vec{X}^a \sigma^a}{r} ; \quad X_4 = i X_0$$
$$\underline{r^2 = X_4^2 + X_1^2 + X_2^2 + X_3^2}$$

$$f(r) = \frac{r^2}{r^2 + \lambda^2}$$

size of the instanton
not fixed classically (scale invariant)

Exercise: Verify that the above solves the self-duality equation.

Real QCD - the strong CP problem

Classical instantons (and anti-instantons) induce vacuum-to-vacuum transitions, which result in θ -vacuum. Next, we need to integrate over quantum fluctuations around instanton configuration

$$Z = \underbrace{e^{-\frac{8\pi^2}{g^2}}}_{\text{anti-instantons}} e^{i\theta} \int D(\mathbb{G}_r^a) Dq D\bar{q} e^{-\frac{S^2}{g^2}} + \text{h.c.}$$

Integration over quark-fields in the massless limit ($m_0, m_q \rightarrow 0$) gives: $\propto \underline{\text{Det } \not{D}}$

$$\not{D} = \gamma^\mu (\partial_\mu + ig \mathbb{G}_\mu^a T^a)$$

Real QCD - the strong CP problem

Note: $\text{Det } \not{D} = 0!$ ($m_u = 0, m_d = 0$)

Because of zero-modes:

$$\not{D} U = 0, \quad U = \frac{\star}{(\star^2 + r^2)^{3/2}} \xi$$

|
constant
spinor

Similar for $\bar{U}, d$ & $\bar{d}$

If a zero mass quark is present instanton transitions from one vacuum to another is blocked $\Rightarrow$ no θ -vacuum!

Real QCD - the strong CP problem

Non-zero Green's functions must contain at least one of each fermion field with a zero-mode:

$$\langle 0 | \bar{u} u \bar{d} d | 0 \rangle \neq 0 \quad \& \text{ also breaks } U(1)_A \text{ by } 2 \text{ units.}$$

Computation of Green's function involves integration over instanton/anti-instanton size and position:

$$\sum_{n_+, n_- = 0}^{\infty} \frac{x^{n_+ + n_-}}{n_+! n_-!} e^{i\theta(n_+ - n_-)} \left(\int d^4x \det(\bar{q}_{L(n)}^a q_{R(n)}^b) \right)^{n_+} \left(\int d^4x \det(\bar{q}_{R(n)}^a q_{L(n)}^b) \right)^{n_-}$$

Real QCD - the strong CP problem

$$= \left[\begin{aligned} & \propto \int_0^{1/\Lambda_{\text{QCD}}} \frac{d\star}{\star} \star^2 e^{-\frac{8\pi^2}{8(\star)}} \sim \Lambda_{\text{QCD}}^{-2} \\ & \text{Use RGE solution:} \\ & e^{-\frac{8\pi^2}{8(\star)}} = (\star \Lambda_{\text{QCD}})^b, \quad b = 11 - \frac{2}{3}n_f = \frac{29}{3} \end{aligned} \right]$$

$$= \exp \left[\int d^4x \underbrace{\mathcal{L}_{\text{QCD}}}_{\mathcal{L}_{\text{Hooft}}} + e^{i\theta} \det(\bar{q}_L^a q_R^b) + e^{-i\theta} \det(\bar{q}_R^a q_L^b) \right]$$

14 Hooft vertex

$$\mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{Hooft}}$$

Real QCD - the strong CP problem

$$\bar{q}_L^a q_R^b = F e^{i \frac{\bar{n} \cdot \vec{\sigma} + n_5 \sigma_5}{f_\pi}}$$

$$\det \bar{q}_L^a q_R^b = F^2 e^{i \frac{2\eta}{f_\pi}}$$

$$\text{Hence, } \mathcal{L}_{\text{Hoot}} = \alpha F^2 \cos\left(\theta + \frac{2\eta}{f_\pi}\right)$$

$$\text{Note: } \langle \eta \rangle = - \frac{f_\pi \theta}{2}$$

$$m_\eta^2 \sim \frac{\alpha F^2}{f_\pi^2} = \left| \alpha \sim F \sim \Lambda_{\text{QCD}} \right| \simeq \frac{\Lambda_{\text{QCD}}^4}{f_\pi^2} \gg m_\pi^2$$

S. Coleman, The uses of instantons, Erice Lectures, 1977

Real QCD - the strong CP problem

Now, let's turn to the problem: We have established existence of CP-violating phase θ in strong interactions due to the instanton mediated transition between topologically distinct vacua

Exercise: Does additional θ -term exist in the electroweak sector? Explain.

- Electric dipole moments (EDMs) are sensitive probe of CP violation

Real QCD - the strong CP problem

Sketch of calculations:

- Perform chiral rotation of quarks and move $\bar{\theta}$ back to mass terms:

$$\bar{\theta} \frac{2g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{\mu\nu a} \longleftrightarrow - \left(\frac{m_u m_d}{m_u + m_d} \right) \bar{\theta} \bar{q}^a i \gamma_5 q^a$$

$\nearrow (\sin \bar{\theta} \simeq \bar{\theta}, \text{ small } \bar{\theta})$

Exercise: Obtain the above term

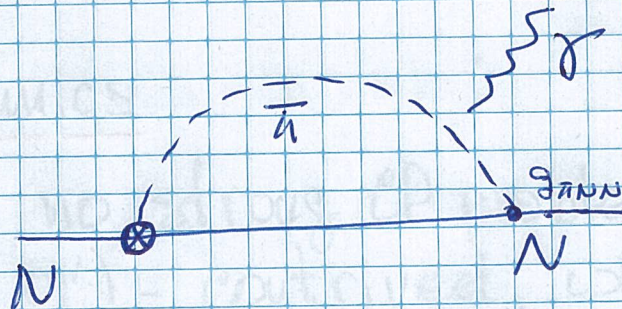
- Coupling to pions: $SU(2)_{L-R}$ transformation:

$$q \rightarrow \left(1 + i \frac{\pi \cdot \sigma}{f_\pi} \right) q \Rightarrow \boxed{\bar{\theta} \frac{1}{f_\pi} \left(\frac{m_u m_d}{m_u + m_d} \right) \bar{q}^a i \gamma_5 q^a}$$

For detailed calculation: Crewther, Di Vecchia, Veneziano, Witten PLB 88 (1979) 23.

Real QCD - the strong CP problem

Then compute



$$d_n = g_{\pi NN} \frac{-\bar{A} m_u m_d}{(m_u + m_d)} \frac{M_N}{4\bar{n}^2} \ln(M_N/m_{\bar{n}}) \langle N_i | \bar{q} \vec{\tau}^i q | N_i \rangle$$

Numerics: $d_n \approx 5 \cdot 10^{-16} \bar{A} \text{ cm}$

Observations: $d_n \lesssim 10^{-25} \text{ cm}$

Hence, $\boxed{\bar{A} = A + \arg(\det M_u M_d) \lesssim 10^{-9}}$

Puzzling cancellation of 2 ~~independent~~ a priori independent phases. The strong CP problem!

Solutions to the strong CP problem

- Non-standard dynamics

E.g. no instantons no strong CP problem (5D ~~not~~ quasi-localised gluons, A.K. Chaudhian 01') - contrived; what about the solution to $U(1)_A$?

- Spontaneous CP-violation (Nelson 83', Barr 84')

Lagrangian is CP invariant $\bar{\theta} = 0$ at tree level.

Spontaneous breaking reintroduce $\bar{\theta}$ radiatively (typically must vanish at 1-loop) $\Rightarrow$ Complicated models, FCNC + experimental data is in agreement with the standard CKM picture

Solutions to the strong CP problem

- An additional chiral symmetry - rotates $\bar{\theta}$ -vacua away

(i) Massless up-quark, $m_u = 0$ (Kaplan, Manohar 86')

Disfavoured by the standard current algebra analyses & lattice calculations - all data seems consistent with $m_u \neq 0$.

(ii) The Standard Model has additional global $U(1)_{PQ}$ chiral symmetry which is spontaneously broken $\Rightarrow$

(pseudo)Goldstone boson, the axion, replaces effectively static CP-violating phase $\bar{\theta}$ with CP-conserving dynamical field. (Dececi, Quinn 77')

Axion - generic consideration

$$U(1)_{PQ} : \frac{a(x)}{f_a} \rightarrow \frac{a(x)}{f_a} + \alpha$$

axion decay constant
phase of $U(1)_{PQ}$ transformations

Low-energy Lagrangian:

$$\mathcal{L} = \mathcal{L}_{SM} + \bar{\Theta} \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} + \frac{1}{2} \partial_\mu a \partial^\mu a +$$

$$+ \mathcal{L}_{int.} \left[\frac{\partial^\mu a}{f_a} ; \psi \right] + \underbrace{\xi \frac{a}{f_a} \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}}$$

$U(1)_{PQ}$ anomaly: $\partial_\mu J_{PQ}^\mu = \xi \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$

So that, $-\frac{\partial_\mu a}{f_a} J_{PQ}^\mu \rightarrow + \frac{a}{f_a} \partial_\mu J_{PQ}^\mu = \xi \frac{a}{f_a} \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$

Axion - general consideration

QCD instantons generate potential for $a(x)$.
Calculations parallel the case of $U(1)_A$:

$$V_a \sim \cos\left(\bar{A} + \frac{a}{f_a}\right) \quad \left| \begin{array}{l} \text{here we omitted} \\ \bar{h}'s, \eta. \end{array} \right.$$
$$\Downarrow$$
$$\langle a \rangle = -\frac{f_a}{\xi} \bar{A}$$

When Lagrangian is written in terms of physical axion field $a_{ph} = a - \langle a \rangle$, CP-violating $\bar{A}$ -term cancels out!

Weak scale axion

- To make SM invariant under an extra chiral $U(1)_{PQ}$ symmetry one must introduce an extra Higgs field:

$$\mathcal{L}_{Yukawa} = Y_{ij}^{(u)} \bar{Q}_{Li} \Phi_1 U_{Rj} + Y_{ij}^{(d)} \bar{Q}_{Li} \Phi_2 d_{Rj} + Y_{ij}^{(e)} \bar{L}_{Li} \Phi_2 e_{Rj}$$

$$|i, j = 1, 2 \dots N_g (= 3)|$$

$$U(1)_{PQ}: U_{Ri} \rightarrow e^{iq_u \alpha} U_{Ri}, \quad d_{Ri} \rightarrow e^{iq_d \alpha} d_{Ri}, \quad e_R \rightarrow e^{iq_e \alpha} e_R$$
$$\Phi_1 \rightarrow e^{-iq_u \alpha}, \quad \Phi_2 \rightarrow e^{-iq_d \alpha}$$

q_u, q_d - $U(1)_{PQ}$ charges

Weak scale axion

$$V(\Phi_1, \Phi_2) = \mu_1^2 (\Phi_1^\dagger \Phi_1) + \mu_2^2 (\Phi_2^\dagger \Phi_2) + \left[\tilde{M} (\Phi_1 \cdot \Phi_2) + \text{h.c.} \right] \\ + \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \left[\delta (\Phi_1 \cdot \Phi_2)^2 + \text{h.c.} \right] \\ + \rho (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2)$$

$\nearrow U(1)_{PQ}$
 $\nearrow U(1)_{PQ}$

The potential is arranged in such a way that

$$\langle 0 | \Phi_1 | 0 \rangle = \frac{v_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle 0 | \Phi_2 | 0 \rangle = \frac{v_2}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Phi_1 = \begin{pmatrix} \Phi_1^+ \\ \Phi_1^0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \Phi_2^0 \\ \Phi_2^- \end{pmatrix}$$

4 real components + 4 real components

4 real massless scalar fields: $SU(2) \times U(1)_Y \times U(1)_{PQ} \rightarrow U(1)_{E_1}$

$$3 + 1 + 1 - 1 = 4$$

3 would be Goldstone bosons + 1 axion

Weak scale axion

Axion must be a massless state which is orthogonal to the weak hypercharge

$$\Phi_1 = \frac{v_1}{\sqrt{2}} \begin{bmatrix} e^{\frac{i g(x)}{v_1}} \\ 0 \end{bmatrix}, \quad \Phi_2 = \frac{v_2}{\sqrt{2}} \begin{bmatrix} 0 \\ e^{\frac{i \tilde{g}(x)}{v_2}} \end{bmatrix}$$

$$v_{EW} = \sqrt{v_1^2 + v_2^2} \approx 246 \text{ GeV}$$

$$\mathcal{L}_{H-V(A)_Y} \propto \left[(\Phi_1^\dagger \partial_\mu \Phi_1 - \partial_\mu \Phi_1^\dagger \cdot \Phi_1) - (\Phi_2^\dagger \partial_\mu \Phi_2 - \partial_\mu \Phi_2^\dagger \cdot \Phi_2) \right] B^\mu$$

$$\propto v_{EW} \partial_\mu \left(\frac{v_1}{v_{EW}} g(x) - \frac{v_2}{v_{EW}} \tilde{g}(x) \right) B^\mu$$

$= G^0$ - would-be Goldstone state

$$\boxed{a(x) = \frac{v_2}{v_{EW}} g + \frac{v_1}{v_{EW}} \tilde{g}} - \text{axion } a(x) \perp G^0(x)$$

Weak scale axion

$$g = \frac{v_2}{v_E} a(x) + \frac{v_1}{v_{EW}} \phi^0(x)$$

$$\tilde{g} = \frac{v_1}{v_E} a(x) - \frac{v_2}{v_{EW}} \phi^0(x)$$

Remove $\phi^0(x)$ by fixing the unitary gauge:

$$\phi_1 = \frac{v_1}{2} \begin{bmatrix} e^{i \frac{\xi a(x)}{v_{EW}}} \\ 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 \\ e^{i \frac{1}{\xi} \frac{a(x)}{v_{EW}}} \end{bmatrix}$$

$$\xi = \frac{v_2}{v_1}$$

Recall, $a(x)$ is a massless field at this point, which survives in the low-energy action of the theory.

Weak scale axion

$U(1)_{PQ}$ is spontaneously broken, however, the low-energy Lagrangian ~~exhibits~~ exhibits PQ shift symmetry:

$$a \rightarrow a + \alpha V_{EW}; \quad \underline{U_{Ri} \rightarrow e^{-i\alpha \frac{1}{3}} U_{Ri}; \quad d_{Ri} \rightarrow e^{-i\alpha \frac{1}{3}} d_{Ri};}$$
$$\underline{e_{Ri} \rightarrow e^{-i\alpha \frac{1}{3}} e_{Ri}}$$

chiral phase
transformations!

Corresponding current

$$\bar{J}_{PQ}^M = -V_{EW} \partial_\mu a + \sum_{i=1}^{N_g} \bar{U}_{iR} \gamma^M U_{iR} + \frac{1}{3} \sum_{i=1}^{N_g} \bar{d}_{Ri} \gamma^M d_{Ri}$$

(leptons are dropped)

This current is anomalous:

$$\partial_\mu \bar{J}_{PQ}^M = A \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$$
$$A = N_g \left(\frac{2}{3} + \frac{1}{3} \right)$$

Weak scale axion

At low energies consider only $N_f = 2$ flavours of quarks:

~~$\frac{1}{2} \begin{pmatrix} u \\ d \end{pmatrix}$~~ , $q = (u, d)^T$

Light meson bound states we again parameterise as:

$$\bar{q}_f^a q_f^b = \vec{\Sigma} = F e^{i \frac{\vec{\pi} \cdot \vec{\sigma} + b \sigma^0}{f_\pi}}$$

Axion-meson interactions can be read off from the Higgs-fermion Yukawa couplings:

$$\mathcal{L}_{a-\bar{n}} = \frac{1}{2} F \text{Tr} \left[\vec{\Sigma} \cdot \hat{\mathbf{A}} \hat{\mathbf{M}} + (\vec{\Sigma} \hat{\mathbf{A}} \cdot \mathbf{M})^+ \right]$$
$$F = \frac{(f_\pi m_{\bar{n}})^2}{m_u + m_d}; \quad \hat{\mathbf{A}} = \begin{bmatrix} e^{i \frac{1}{3} \frac{a(x)}{f_{\text{EW}}}} & 0 \\ 0 & e^{i \frac{1}{3} \frac{a(x)}{f_{\text{EW}}}} \end{bmatrix}, \quad \hat{\mathbf{M}} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

Weak scale axion

Expand and keep only quadratic terms in neutral mesons π^0, η & a

$$\mathcal{L}_{a-\pi}^{(2)} = \frac{m_{\pi}^2}{2} \left[\frac{m_u}{m_u + m_d} \left[\pi^0 + \eta - \frac{3f_{\pi}}{5\sigma_{EW}} a \right]^2 + \frac{m_d}{m_u + m_d} \left[\eta - \pi^0 - \frac{f_{\pi}}{5\sigma_{EW}} a \right]^2 \right]$$

$$\boxed{M_a = 0, \quad m_{\pi} \approx m_{\eta}}$$

Exercise: Verify this

Now let's add instanton induced anomaly term:

$$\mathcal{L}_{anomaly} = \frac{m_{\eta}^2}{2} \left[\eta + \frac{f_{\pi}}{\sigma_{EW}} \frac{(N_f - 1)(\frac{1}{3} + \frac{1}{3})}{2} a \right]^2$$

only heavy quarks contribute

relative $U(1)_A$ & $U(1)_{PR}$ anomaly coefficients

Weak scale anomaly

Exercise: Diagonalise mass matrix for π^0 , η & a

$$\mathcal{L}_{a-\pi}^{(2)} + \mathcal{L}_{\text{anomaly}}$$

$$M_a = \lambda_m \frac{f_{\bar{u}} m_{\bar{u}}}{v_{EW}} \frac{\sqrt{m_u m_d}}{m_u + m_d} = \lambda_m \cdot 25 \text{ keV} \quad \text{axion mass}$$

$$\alpha_{a\pi} = \lambda_3 \frac{f_{\bar{u}}}{v_{EW}} \quad \text{— axion-pion mixing parameter}$$

$$\alpha_{a\eta} = \lambda_0 \frac{f_{\bar{u}}}{v_{EW}} \quad \text{— axion-}\eta \text{ mixing}$$

$$\lambda_m = N_g \left(\frac{2}{3} + \frac{1}{3} \right); \quad \lambda_3 = \frac{1}{2} \left[\left(\frac{2}{3} - \frac{1}{3} \right) - N_g \left(\frac{2}{3} + \frac{1}{3} \right) \frac{m_d - m_u}{m_d + m_u} \right]$$

$$\lambda_0 = \frac{1}{2} (1 - N_g) \left(\frac{2}{3} + \frac{1}{3} \right)$$

Note a & π^0 are pseudo Goldstone bosons, i.e. $M_{a,\pi} \xrightarrow{m_{u,d} \rightarrow 0} 0$,
while η receives mass from anomaly: $M_{\eta} \xrightarrow{m_{\text{quark}} \neq 0} \neq 0$

Weak scale axion

$$a \simeq a_{ph} - \frac{1}{f_\pi} \bar{\pi}_{ph} - \frac{1}{f_\pi} h_{ph}$$

Axion couplings can be deduced from $\bar{\pi}$ & h couplings employing a - $\bar{\pi}$ - h mixing.

$$\text{E.g., } \mathcal{L}_{\bar{\pi}, h} = \frac{\alpha}{4\pi} \left[\frac{\bar{\pi}^0}{f_\pi} + \frac{5}{3} \frac{h}{f_\pi} \right] F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$\bar{\pi}, h$ - photon coupling.

mixing gives a interaction with photons induced by light quarks. Adding heavy quarks to this one obtains:

$$\mathcal{L}_{ax\gamma} = \frac{\alpha}{4\pi} K_{ax\gamma} \frac{a_{ph}}{v_{EW}} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$K_{ax\gamma} = N_g \left(\frac{2}{3} + \frac{1}{3} \right) \frac{m_u}{m_u + m_d}$$

Weak scale axion

The weak scale axion is ~~light~~ light, $m_a = \lambda_m \cdot 25 \text{ keV} < 2m_e$ (unless $\lambda_m \gg 1$). Hence, it's also long-lived since only decay $a \rightarrow 2\gamma$.

• Such long lived axions are excluded by observations:

E.g. $\text{Br}(K^+ \rightarrow \pi^+ a) \simeq 3 \cdot 10^{-5} \lambda_0^2$ (Theory)

KEK bound: $\text{Br}(K^+ \rightarrow \bar{a}^+ \text{ nothing}) < 3.8 \cdot 10^{-8}$ (experiment)

• If $\lambda_0 = 0$ (e.g. only $N_8 = 1$ is charged under $U(1)_{PQ}$), the weak scale long lived axion is still excluded from non-observation of axion production via nuclear de-excitation in reactors (sensitive to λ_3 as well).

• Short lived axion $m_a > 2m_e$ requires ξ or $1/\xi$ large
 $\Gamma(\psi \rightarrow a\gamma) \sim \xi^2$ & $\Gamma(\psi \rightarrow a\gamma) \sim 1/\xi^2$ — hence, excluded!

Invisible axion

If $U(1)_{PQ}$ is spontaneously broken at much higher scale

$$U_{EW} \rightarrow f_a \gg U_{EW}$$

then (i) it is much lighter

(ii) much more feebly interacts with ordinary matter

We can avoid constraints $\rightarrow$ "Invisible" axion

Invisible axion

① KSVF axion (Kim 79'; Shifman, Vainshtein, Zakharov 80')

Model: add to the Standard Model a singlet scalar ϕ with $\langle \phi \rangle = f_a$ breaking $U(1)_{PQ}$ and heavy quark Q , $M_Q \sim f_a$, which also carries $U(1)_{PQ}$ charge

$$\mathcal{L}_{\text{axion}}^{\text{KSVZ}} = \frac{a}{f_a} \left[\underbrace{\frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}}_{\text{QCD } U(1)_{PQ} \text{ anomaly}} + \underbrace{3e_Q^2 \frac{\alpha}{4\pi} F_{\mu\nu} \tilde{F}^{\mu\nu}}_{\substack{U(1)_A \text{ electromagnetic} \\ \text{anomaly} \\ \text{due to the } Q \text{ quark}}} \right]$$

Axion parameters:

$$M_a = \frac{v_{EW}}{f_a} \lambda_m \left(m_{\bar{n}} \underbrace{\frac{f_{\bar{n}}}{v_{EW}} \frac{\sqrt{m_u m_d}}{m_u + m_d}}_{25 \text{ keV}} \right)$$

$$\alpha_{a\bar{n}} = \lambda_3 \frac{f_{\bar{n}}}{f_a}; \quad \alpha_{a\eta} = \lambda_0 \frac{f_{\eta}}{f_a}; \quad \lambda_0 = -1/2, \lambda_m = 1, \lambda_3 = -\frac{m_d - m_u}{2(m_u + m_d)}$$

Invisible axion

Photon-axion coupling: $K_{\alpha\gamma\gamma} = 3e_Q^2 - \frac{4m_d + m_u}{3(m_u + m_d)}$

• DFSZ (Dine, Fischler & Srednicki 81'; Zhitnitsky 80')

Model: Original PQ model + a complex electroweak singlet scalar, ϕ , $\langle \phi \rangle = f_a \gg v_{EW}$

Axion parameters: Define $X_1 = \frac{2g_1^2}{v_{EW}^2}$, $X_2 = \frac{2g_2^2}{v_{EW}^2}$

$$M_a^{DFSZ} = M_a^{KSUZ} \left(f_a \rightarrow \frac{f_a}{2N_g} \right)$$

$$\lambda_3 = \frac{1}{2} \left(\frac{X_1 - X_2}{2N_g} - \frac{m_d - m_u}{m_d + m_u} \right), \quad \lambda_0 = \frac{1 - N_g}{2N_g}$$

$$K_{\alpha\gamma\gamma} = \frac{4}{3} - \frac{4m_d + m_u}{3(m_d + m_u)}$$

Astrophysics & Cosmology of axions

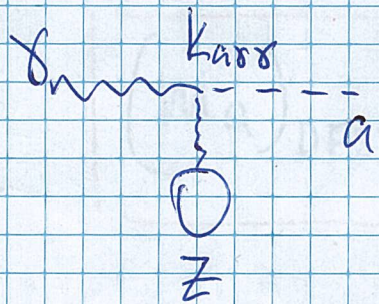
The solution to the strong CP problem seems to predict a light pseudoscalar particle, the axion, which is feebly interacts with matter. Nevertheless, axions have interesting astrophysical & cosmological implications.

- ① The astrophysical bounds on the mass of axion
 m_a (and thus on f_a) ~~is~~ comes from the excessive cooling of stars because of axion emission

The dominant processes: (i) $\gamma e \rightarrow e a$ - Compton axion production
(not applicable to $K S \neq Z$)
Axion - e coupling is involved

Astrophysics & Cosmology of axions

(2) $\gamma Z_n, e \rightarrow Z_n, e a$ Primakoff effect



Energy loss:
Compton process

$$Q = \frac{1}{E_{\text{star}}} \int \underbrace{dn_e}_{\text{stellar density}} \underbrace{dn_\gamma}_{\text{density of initial states}} \underbrace{|\mathcal{M}|^2}_{\text{interaction rate}} E_a$$

axion energy

For $m_a \ll T_{\text{star}} \sim \mathcal{O}(\text{keV})$, the only dependence on m_a comes from $|\mathcal{M}|^2 \propto f^{-2} \sim m_a^2$. One requires $Q \lesssim Q_{\text{nuc}} \sim 10^2 \frac{\text{ergs}}{\text{gr sec}}$
rate of nuclear energy degradation

Astrophysics & Cosmology of axions

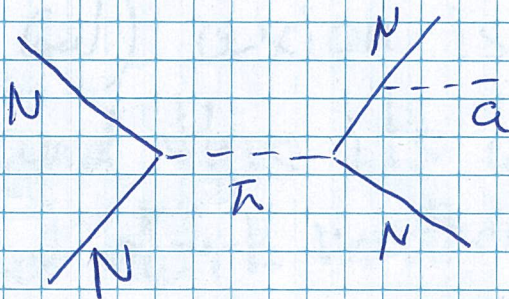
Star cooling (Raffelt 90')

$$(m_a)_{\text{DFSZ}} \leq \frac{10^{-2}}{X_2} \text{ eV}, \quad (m_a)_{\text{KSVZ}} \leq \frac{0.27}{K_{\text{arr}}} \text{ eV}$$

⊙ Bounds from the observation of ν 's from SN 1987a.

If axion luminosity $\sim$ neutrino luminosity ($\sim 10^{53}$ ergs/sec) during the core collapse, then neutrino signal would be altered

Dominant emission:



(Turner 88')

$$(m_a)_{\text{DFSZ}} \leq \frac{1.7 \cdot 10^{-3}}{\xi_{\text{brem}}^2} \text{ eV}$$

$$(m_a)_{\text{KSVZ}} \leq 8.4 \cdot 10^{-4} \text{ eV}$$

$$\xi_{\text{brem}} = 1.44 + \frac{1}{2} (X_1 + X_2 - 1.55)^2$$

Astrophysics & Cosmology of axions

While astrophysics provides with upper bounds on m_a , cosmology constraints the axion mass from below.

The basic phenomenon: Axion starts to oscillate at $T^* \sim \Lambda_{\text{QCD}}$, when QCD effects create a curvature (mass) in its potential. Oscillations of zero-momentum axion field contributes to the Universe's energy density. ~~as~~ ~~in~~ In fact, if f_a is sufficiently large ($f_a \gtrsim 10^{12} \text{ GeV}$) axion is stable within the Universe life and thus its coherent oscillations represent cold dark matter!

Astrophysics & Cosmology of axions

One must make sure that energy of axion oscillations does not exceed the ~~ever~~ critical energy:

$$\rho_{\text{axion}} < \rho_{\text{cr}} = \frac{3H_0^2}{8\pi G_N}$$

• Evolution of axion expectation value $\langle a \rangle$:

$$\frac{d^2 \langle a \rangle}{dt^2} + 3 \frac{\dot{R}}{R} \frac{d \langle a \rangle}{dt} + m^2(t) \langle a \rangle = 0$$

friction

$$\left| \frac{\dot{R}}{R} = H \right.$$

term due to the expansion

Astrophysics & Cosmology of axions

Oscillations start when:

$$M_a(T^*) \sim H(T^*) \sim \frac{T^{*2} \sim \Lambda_{QCD}^2}{M_P} \quad \text{Planck mass}$$

At the start of oscillations $\rho_a(T^*) \sim M_a^2 H^2 \sim M_a^2 f^2$

For slowly changing $M^2(t)$, $\rho_a(t) \sim M_a(t)/R^3(t)$

$$\rho_{\text{axion}} = \rho_a(T^*) \frac{M_a}{M_a(T^*)} \frac{R^3(T^*)}{R^3} \sim \frac{\Lambda_{QCD}^3 T_0^3}{M_a M_P}$$

$T_0 \approx 2.72 \text{ K}$ — temperature today

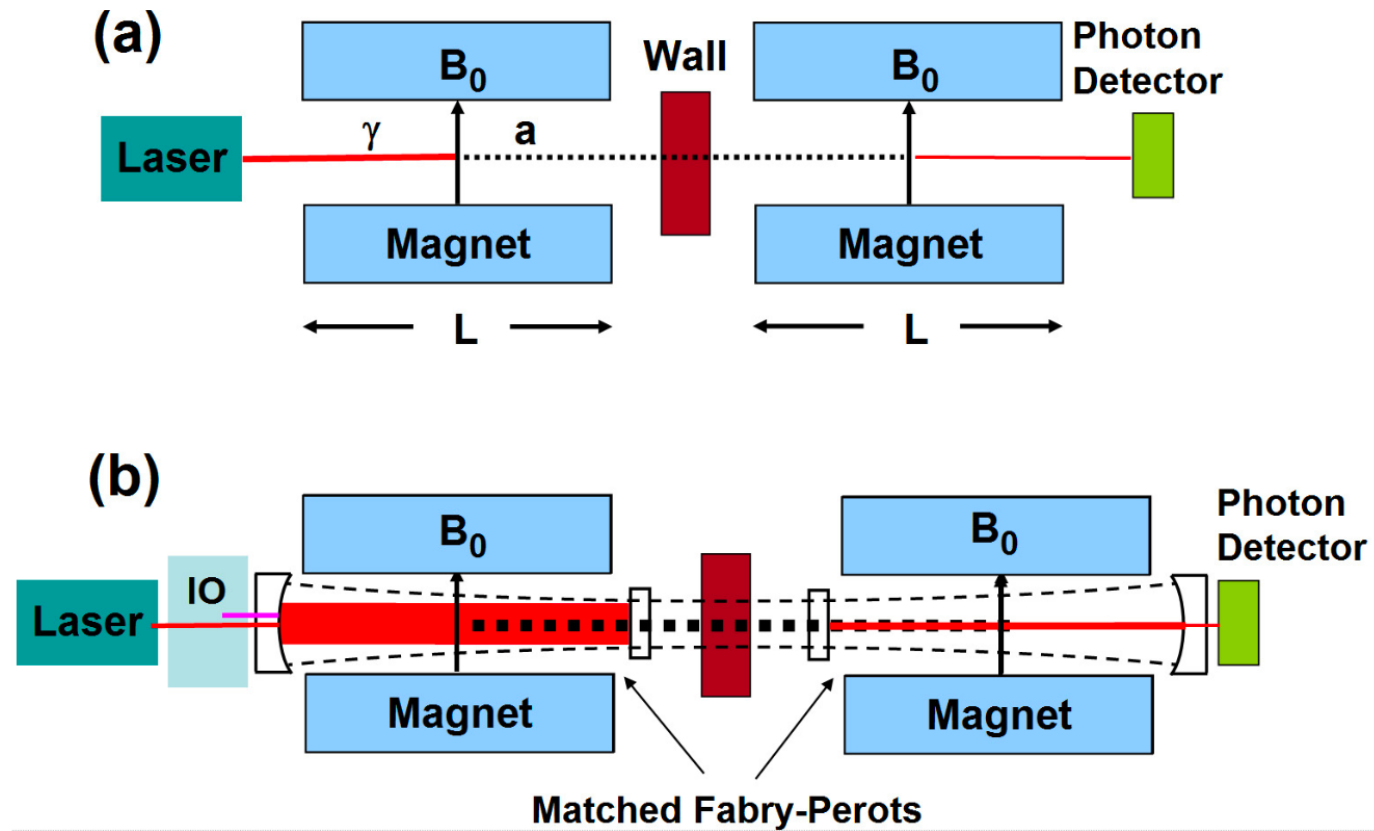
$$M_a \geq 10^{-6} \text{ eV}$$

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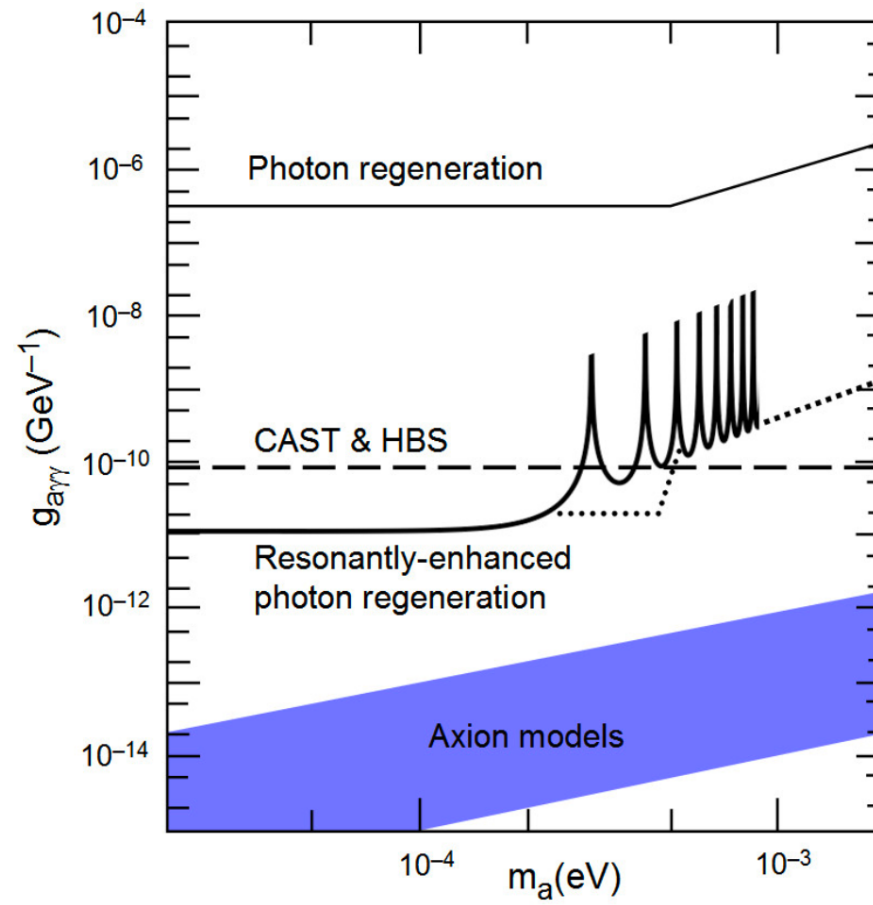
$$10^{-6} \text{ eV} \leq m_a \leq 10^{-3} \text{ eV}$$

Missalignment during inflation (Linde 88') $\langle a \rangle = cf, c \ll 1$
would decrease the lower bound by $\sim c^2$.

Photon regeneration experiments, arXiv:1311.0029



Photon regeneration experiments, arXiv:1311.0029



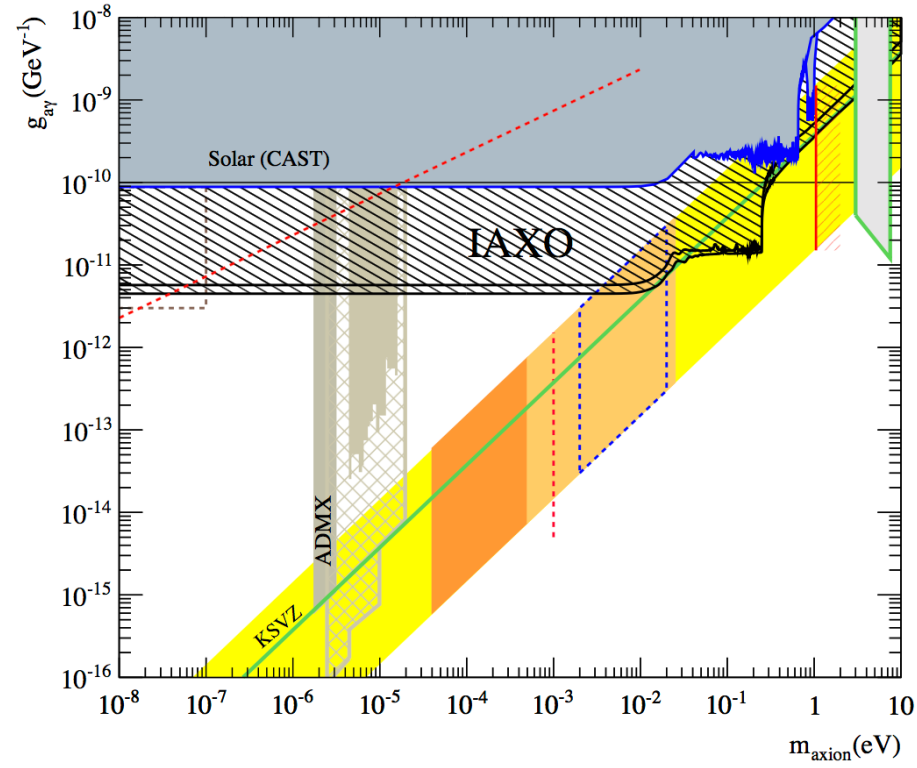
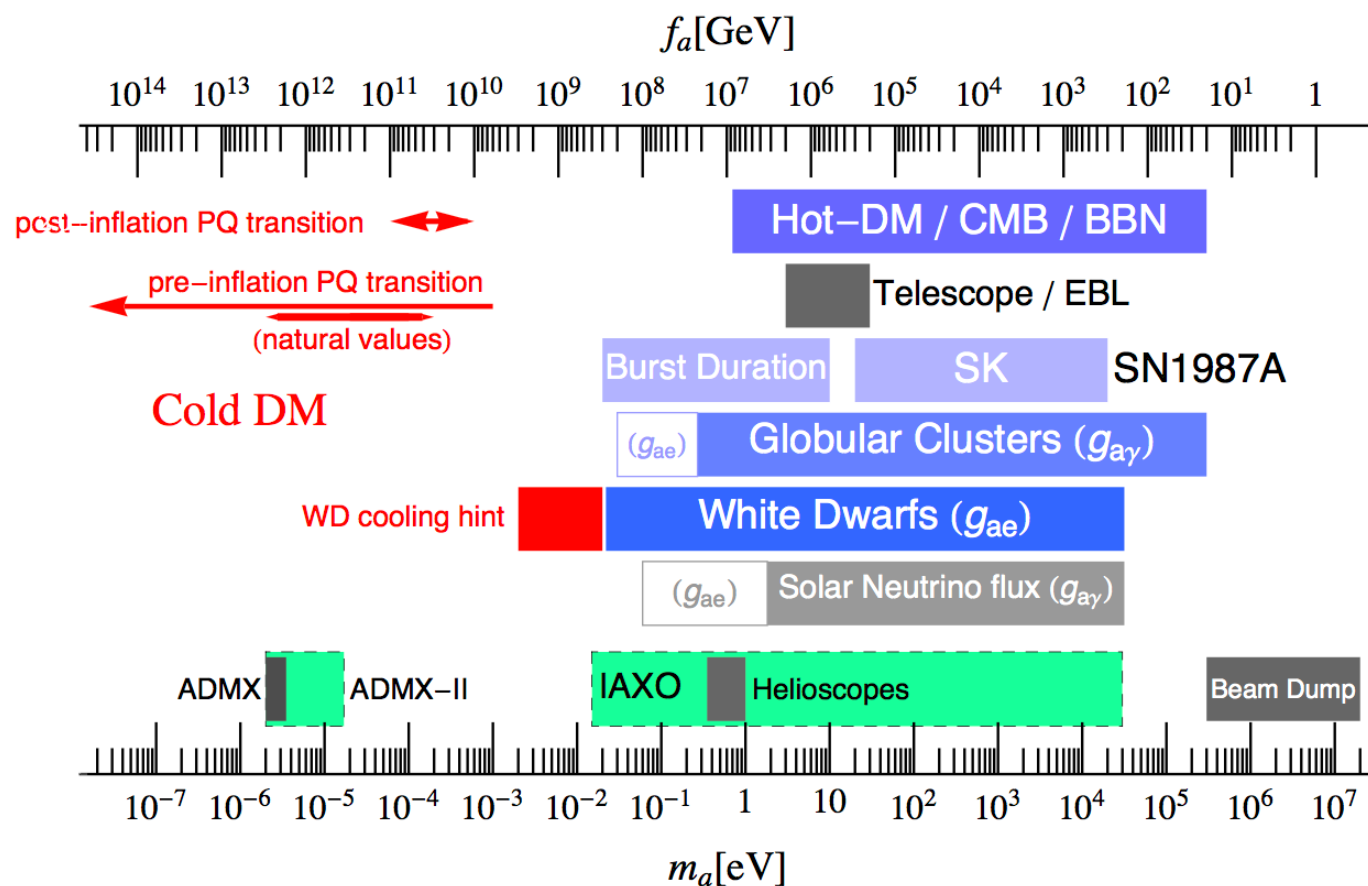


FIG. 5. Expected sensitivity of IAXO compared with current bounds from CAST and ADMX. Also future prospects of ADMX are shown (dashed brown region).

Summary of constraints on axions and searches of axions, arXiv:1311.0029



Summary of constraints on axion-like particles (ALPs), arXiv:1311.0029

