Spin multiplets of supersymmetric mechanics

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Introduction

The standard $\mathcal{N}=4$, $d=1$ superalgebra:

$$\left\{Q^i_\alpha, Q^\beta_j\right\} = 2\delta^i_j\delta^\beta_\alpha H. \tag{1}$$

Supercharges Q^i_α carry fundamental indices ($i=1,2$ and $\alpha=1,2$) of the automorphism group $\mathrm{SO}(4) \sim \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$. The superspace is

$$\zeta := \left\{t, \theta^{i\alpha}\right\}, \tag{2}$$

and transforms as

$$\delta\theta^{i\alpha} = \epsilon^{i\alpha}, \qquad \delta t = -\, i\epsilon^{i\alpha}\theta_{i\alpha}\,, \qquad \overline{(\theta^{i\alpha})} = -\,\theta_{i\alpha}\,, \qquad \overline{(\epsilon^{i\alpha})} = -\,\epsilon_{i\alpha}\,. \tag{3}$$

The covariant derivatives are

$$D^{i\alpha} = \frac{\partial}{\partial\theta_{i\alpha}} + i\theta^{i\alpha}\partial_t\,. \tag{4}$$

Multiplets of $\mathcal{N}=4$, $d=1$ supersymmetric mechanics are denoted as $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ with $\mathbf{k}=\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$. These numbers correspond to the numbers of bosonic physical fields, fermionic physical fields and bosonic auxiliary fields.

Ordinary and mirror multiplets

- The ordinary $\mathcal{N}=4$ multiplets have their mirror counterparts characterized by the interchange of two $\mathrm{SU}(2)$ groups which form $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ automorphism group of the standard $\mathcal{N}=4$ supersymmetry (E. Ivanov, J. Niederle, Phys. Rev. D **80** (2009) 065027).

- Since this interchange $(i, j \longleftrightarrow \alpha, \beta)$ has no essential impact on $\mathcal{N}=4$ supersymmetry, $\mathcal{N}=4$ multiplets and their mirror counterparts are mutually equivalent.

- For example, the ordinary multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ is described by a scalar superfield $\mathcal{X}$ satisfying the quadratic constraint:

$$D_{k(\alpha} D^{k}_{\beta)} \mathcal{X} = 0. \tag{5}$$

For the mirror multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ the quadratic constraint is written as

$$D^{(i}_{\gamma} D^{j)\gamma} X = 0. \tag{6}$$

Semi-dynamical (spin) multiplets

Wess-Zumino (WZ) type Lagrangians for the ordinary multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$ were presented in the framework of the $\mathcal{N}=4$, $d=1$ harmonic superspace (E. Ivanov, O. Lechtenfeld, JHEP **0309** (2003) 073). For example, the simplest WZ Lagrangian for $(\mathbf{4},\mathbf{4},\mathbf{0})$ reads

$$\mathcal{L}_{\rm WZ} = \frac{i}{2}\left(z^i\dot{\bar{z}}_i - \dot{z}^i\bar{z}_i\right) + \psi^a\bar{\psi}_a\,, \qquad i=1,2, \qquad a=1,2. \tag{7}$$

Without kinetic Lagrangian containing second-order in time derivatives of bosonic terms, this Lagrangian describes a semi-dynamical multiplet. Fermionic fields become auxiliary, while the bosonic term produces the primary constraints

$$p_i + \frac{i}{2}\,\bar{z}_i \approx 0, \qquad \bar{p}^j - \frac{i}{2}\,z^j \approx 0. \tag{8}$$

These constraints are second class and we are led to introduce Dirac brackets:

$$\left\{z^i, \bar{z}_j\right\} = i\,\delta^i_j\,. \tag{9}$$

Thus, the bosonic fields z^i, $\bar{z}_j$ describe semi-dynamical degrees of freedom (or spin variables).

Coupling

- Coupling of dynamical and semi-dynamical multiplets was proposed by S. Fedoruk, E. Ivanov, O. Lechtenfeld, Phys. Rev. D **79** (2009) 105015. This idea provided harmonic superfield construction of $\mathcal{N}=4$ extension of Calogero system with the additional spin (isospin) degrees of freedom z^i, $\bar{z}_j$.
- This work was followed by a further study of "spinning" models considering couplings of dynamical and semi-dynamical multiplets (S. Bellucci, S. Krivonos, A. Sutulin, Phys. Rev. D **81** (2010) 105026, E. Ivanov, M. Konyushikhin, A. Smilga, JHEP **1005** (2010) 033, etc).
- The ordinary multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as a semi-dynamical multiplet interacting with the dynamical multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ was considered by S. Fedoruk, E. Ivanov, O. Lechtenfeld, JHEP **1206** (2012) 147. They showed that the triplet of spin variables v^{ij} describes a 2 dimensional surface in $\mathbb{R}^3$ satisfying the so-called Nahm equations.

Plan of talk

- The main goal of the present talk is to consider the interaction of the semi-dynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ with dynamical mirror multiplets (coupling of ordinary dynamical and mirror semi-dynamical multiplets is under question).

- As an instructive example we consider the simplest coupling with the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$. In fact we reproduce the model constructed in S. Fedoruk, E. Ivanov, O. Lechtenfeld, JHEP **1206** (2012) 147, but in terms of mirror superfields in harmonic superspace.

- As new results we present the coupling with the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ which belongs to the mirror classification of $\mathcal{N}=4$ multiplets. The corresponding interaction is constructed as a superpotential in the chiral subspace.

Mirror multiplets

We define mirror multiplets by superfields carrying no external i, j indices and satisfying the common constraint

$$D^{(i}_{\gamma} D^{j)\gamma} M = 0. \tag{10}$$

The list of mirror multiplets:

- $(\mathbf{0}, \mathbf{4}, \mathbf{4})$: $D^{i(\alpha}\Psi^{\beta)A} = 0, \qquad \overline{(\Psi^{\alpha A})} = \Psi_{\alpha A}\,, \qquad A = 1, 2.$
- $(\mathbf{1}, \mathbf{4}, \mathbf{3})$: $D^{(i}_{\alpha} D^{j)\alpha} X = 0.$
- $(\mathbf{2}, \mathbf{4}, \mathbf{2})$: $D^{i2} Z = \bar{D}^i Z = 0, \qquad D^{i1} \bar{Z} = D^i \bar{Z} = 0.$
- $(\mathbf{3}, \mathbf{4}, \mathbf{1})$: $D^{i(\alpha} V^{\beta\gamma)} = 0, \qquad V^{\alpha\beta} = V^{\beta\alpha}, \qquad \overline{(V^{\alpha\beta})} = -\, V_{\alpha\beta}\,.$
- $(\mathbf{4}, \mathbf{4}, \mathbf{0})$: $D^{i(\alpha} Y^{\beta)A} = 0, \qquad \overline{(Y^{\alpha A})} = Y_{\alpha A}\,, \qquad A = 1, 2.$

The multiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ is described by a fermionic superfield $\Psi^{\alpha A}$. The rest of multiplets are described by bosonic superfields.

Harmonic superspace

Listed mirror multiplets have a description in the standard $\mathcal{N}=4$, $d=1$ harmonic superspace

$$\zeta_{\rm H} := \left\{t_{(\rm A)}, \theta^{\pm}_{\alpha}, u^{\pm}_{i}\right\}, \tag{11}$$

where

$$t_{(\rm A)} = t - \frac{i}{2}\,\theta^{i}_{\alpha}\theta^{j\alpha}\left(u^{+}_{i}u^{-}_{j} + u^{+}_{j}u^{-}_{i}\right), \qquad \theta^{\pm}_{\alpha} := \theta^{i}_{\alpha}u^{\pm}_{i}, \qquad u^{+}_{i}u^{-}_{j} - u^{+}_{j}u^{-}_{i} = \varepsilon_{ij}\,. \tag{12}$$

Covariant derivatives are defined as

$$D^{+\alpha} = \frac{\partial}{\partial\theta^{-}_{\alpha}}\,, \qquad D^{++} = \partial^{++} - i\theta^{+}_{\alpha}\theta^{+\alpha}\frac{\partial}{\partial t_{(\rm A)}} + \theta^{+}_{\alpha}\frac{\partial}{\partial\theta^{-}_{\alpha}}\,, \qquad D^{0} = \partial^{0} + \theta^{+}_{\alpha}\frac{\partial}{\partial\theta^{+}_{\alpha}} - \theta^{-}_{\alpha}\frac{\partial}{\partial\theta^{-}_{\alpha}}\,, \tag{13}$$

where the partial harmonic derivatives are

$$\partial^{++} := u^{+}_{i}\frac{\partial}{\partial u^{-}_{i}}\,, \qquad \partial^{0} := u^{+}_{i}\frac{\partial}{\partial u^{+}_{i}} - u^{-}_{i}\frac{\partial}{\partial u^{-}_{i}}\,. \tag{14}$$

The harmonic superspace contains the analytic harmonic subspace parametrized by the reduced coordinate set

$$\zeta_{(\rm A)} := \left\{t_{(\rm A)}, \theta^{+}_{\alpha}, u^{\pm}_{i}\right\}, \qquad D^{+\alpha}\zeta_{(\rm A)} = 0, \tag{15}$$

which is closed under $\mathcal{N}=4$ supersymmetry.

Mirror multiplets in harmonic superspace

Mirror multiplets are described by neutral harmonic superfields satisfying

$$D^{++}M = 0, \qquad D^0 M = 0, \qquad D^+_\alpha D^{+\alpha} M = 0. \tag{16}$$

Harmonic constraints of mirror multiplets:

- $(\mathbf{0},\mathbf{4},\mathbf{4})$: $\quad D^{+(\alpha}\Psi^{\beta)A} = 0, \qquad D^{++}\Psi^{\alpha A} = 0.$
- $(\mathbf{1},\mathbf{4},\mathbf{3})$: $\quad D^+_\alpha D^{+\alpha} X = 0, \qquad D^{++} X = 0.$
- $(\mathbf{2},\mathbf{4},\mathbf{2})$: $\quad D^{+2} Z = 0, \qquad D^{+1}\bar{Z} = 0, \qquad D^{++}Z = 0, \qquad D^{++}\bar{Z} = 0.$
- $(\mathbf{3},\mathbf{4},\mathbf{1})$: $\quad D^{+(\alpha} V^{\beta\gamma)} = 0, \qquad D^{++}V^{\alpha\beta} = 0.$
- $(\mathbf{4},\mathbf{4},\mathbf{0})$: $\quad D^{+(\alpha} Y^{\beta)A} = 0, \qquad D^{++}Y^{\alpha A} = 0.$

Mirror multiplet (3,4,1)

The mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is described by a triplet superfield $V^{\alpha\beta}$ satisfying

$$D^{+(\alpha}V^{\beta\gamma)} = 0, \qquad D^{++}V^{\alpha\beta} = 0, \qquad V^{\alpha\beta} = V^{\beta\alpha}, \qquad \overline{(V^{\alpha\beta})} = -\,V_{\alpha\beta}\,. \tag{17}$$

The solution is

$$V^{\alpha\beta} = v^{\alpha\beta} + \theta^{-(\alpha}\chi^{i\beta)}u_i^+ - \theta^{+(\alpha}\chi^{i\beta)}u_i^- - 2i\,\theta^{-(\alpha}\theta^+_\gamma\dot{v}^{\beta)\gamma} + \theta^{-(\alpha}\theta^{+\beta)}C - i\,\theta^{+\gamma}\theta^+_\gamma\theta^{-(\alpha}\dot{\chi}^{i\beta)}u_i^-\,, \tag{18}$$

where

$$\overline{(v^{\alpha\beta})} = -\,v_{\alpha\beta}\,, \qquad \overline{(\chi^{k\alpha})} = -\,\chi_{k\alpha}\,, \qquad \overline{(C)} = C. \tag{19}$$

Component fields transform as

$$\delta v^{\alpha\beta} = \epsilon^{i(\alpha}\chi_i^{\beta)}, \qquad \delta\chi^{i\alpha} = 2i\epsilon^i_\beta\dot{v}^{\alpha\beta} - \epsilon^{i\alpha}C, \qquad \delta C = -\,i\epsilon_{i\alpha}\dot{\chi}^{i\alpha}. \tag{20}$$

Harmonic analytic integrals

The ordinary multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is described by the analytic superfield $\mathcal{V}^{++}$ satisfying

$$D^{+\alpha}\mathcal{V}^{++}=0, \qquad D^{++}\mathcal{V}^{++}=0. \tag{21}$$

The corresponding WZ action was defined as the integral over the analytic superspace (E. Ivanov, O. Lechtenfeld, JHEP **0309** (2003) 073):

$$S'_{\rm WZ}=\int d\zeta^{--}_{(\rm A)}\,\mathcal{L}^{++}\left(\mathcal{V}^{++},u^{\pm}_i\right), \qquad D^{+\alpha}\mathcal{L}^{++}\left(\mathcal{V}^{++},u^{\pm}_i\right)=0. \tag{22}$$

This analytic superpotential is manifestly $\mathcal{N}=4$ supersymmetric since the Lagrangian is defined on the analytic superspace and the integral is taken over this superspace.

Alternative construction

Here we consider an alternative construction of WZ action for mirror multiplets in the same analytic superspace $\zeta_{(\mathrm{A})}$. Since mirror superfields carry no external charges $\pm$, we must compensate the charge -2 of the analytic measure $d\zeta^{--}_{(\mathrm{A})}$ by covariant derivatives $D^{\pm\alpha}$ and superspace coordinates $\theta^{\pm}_{\alpha}$. We choose the following simplest ansatz:

$$S_{\mathrm{WZ}} = \int d\zeta^{--}_{(\mathrm{A})}\,\theta^{+}_{\alpha} D^{+}_{\beta} L^{\alpha\beta}(V), \qquad L^{\alpha\beta}(V) = L^{\beta\alpha}(V). \tag{23}$$

We must require that the integrand satisfies the analyticity condition that yields

$$D^{+\gamma}\left[\theta^{+}_{\alpha} D^{+}_{\beta} L^{\alpha\beta}(V)\right] = 0 \quad \Rightarrow \quad D^{+}_{\gamma} D^{+\gamma} L^{\alpha\beta}(V) = 0. \tag{24}$$

The quadratic constraint is the necessary analyticity condition that preserves the invariance of the WZ action:

$$\delta S_{\mathrm{WZ}} = \int d\zeta^{--}_{(\mathrm{A})}\,\epsilon^{+}_{\alpha} D^{+}_{\beta} L^{\alpha\beta} = \int d\zeta^{--}_{(\mathrm{A})}\,D^{++}\left(\epsilon^{-}_{\alpha} D^{+}_{\beta} L^{\alpha\beta}\right) = 0 \quad \Rightarrow$$

$$\Rightarrow \quad D^{+\gamma}\left(\epsilon^{-}_{\alpha} D^{+}_{\beta} L^{\alpha\beta}\right) = \frac{1}{2}\,\epsilon^{-}_{\alpha} D^{+}_{\beta} D^{+\beta} L^{\alpha\gamma} = 0. \tag{25}$$

Wess-Zumino Lagrangian

Component Lagrangian reads

$$\mathcal{L}_{\rm WZ} = C\,\mathcal{U} + i\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta} + \frac{1}{2}\,\mathcal{R}^{\alpha\beta}\chi^{i}_{\alpha}\chi_{i\beta}\,, \tag{26}$$

where

$$\mathcal{U} = \partial^{\alpha\beta}L_{\alpha\beta}\,, \qquad \mathcal{A}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\,\partial^{\gamma\delta}L_{\beta\delta} + \varepsilon_{\beta\gamma}\,\partial^{\gamma\delta}L_{\alpha\delta}\,, \qquad \mathcal{R}^{\alpha\beta} = \partial^{\alpha\gamma}\partial^{\beta\delta}L_{\gamma\delta}\,. \tag{27}$$

One can check that

$$\partial_{\alpha\beta}\,\mathcal{U} = \mathcal{R}_{\alpha\beta}\,, \qquad \Delta_3\,\mathcal{U} = 0, \qquad \partial^{\alpha\beta}\mathcal{A}_{\alpha\beta} = 0, \qquad \partial_{\alpha\beta}\,\mathcal{A}_{\gamma\delta} - \partial_{\gamma\delta}\,\mathcal{A}_{\alpha\beta} = \varepsilon_{\alpha\delta}\,\mathcal{R}_{\beta\gamma} + \varepsilon_{\beta\gamma}\,\mathcal{R}_{\alpha\delta}\,. \tag{28}$$

Fermionic fields are excluded by their equations of motion. Constraints of the system are

$$\pi_{\alpha\beta} = p_{\alpha\beta} - i\,\mathcal{A}_{\alpha\beta} \approx 0, \qquad \mathcal{U} \approx 0. \tag{29}$$

The last constraint appears as a secondary one from the primary constraint $p_C \approx 0$.

Spin variables

The constraint $\mathcal{U} \approx 0$ kills one degree of freedom of the triplet $v^{\alpha\beta}$, so the triplet describes 2 dimensional surface in $\mathbb{R}^3$. The matrix formed by Poisson brackets of the constraints is not degenerate:

$$\det \begin{vmatrix} \{\pi_{\alpha\beta}, \pi_{\gamma\delta}\}_{\rm PB} & \{\pi_{\alpha\beta}, \mathcal{U}\}_{\rm PB} \\ \{\mathcal{U}, \pi_{\gamma\delta}\}_{\rm PB} & 0 \end{vmatrix} \neq 0. \tag{30}$$

Calculating the inverse matrix we find the corresponding Dirac brackets

$$\{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i\left(\varepsilon_{\alpha\gamma}\mathcal{R}_{\beta\delta} + \varepsilon_{\beta\delta}\mathcal{R}_{\alpha\gamma}\right)}{2\,\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}}\,. \tag{31}$$

Non-commutative plane

Let us consider the simplest case $\mathcal{U} \sim y$ when the Lagrangian is written as

$$\mathcal{L}_{\rm WZ} = \frac{i}{2}\left(u\dot{\bar{u}} - \dot{u}\bar{u}\right) + \frac{C}{2}\left(c - y\right) - \frac{1}{4}\,\chi^i_1\chi_{i2}\,, \tag{32}$$

where

$$v_{12} = y, \qquad v_{11} = -\sqrt{2}\,u, \qquad v_{22} = \sqrt{2}\,\bar{u}. \tag{33}$$

The matrix takes on a very simple and non-degenerate form

$$\begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{vmatrix}. \tag{34}$$

Dirac brackets are

$$\{u, \bar{u}\} = i, \qquad \{y, \bar{u}\} = 0, \qquad \{y, \bar{u}\} = 0. \tag{35}$$

The complex field u describes a non-commutative plane in $\mathbb{R}^3$, while the third coordinate (component) y, perpendicular to this plane, takes on a constant value $y = c$.

Relation to fuzzy sphere

In S. Fedoruk, E. Ivanov, O. Lechtenfeld, JHEP **1206** (2012) 147 only fuzzy sphere solution was given as a solution of the 3 dimensional Laplace equation:

$$\mathcal{U} \sim \frac{1}{\sqrt{y^2 + 2u\bar{u}}}, \qquad \left(\partial_y^2 + 2\,\partial_u \partial_{\bar{u}}\right)\mathcal{U} = 0. \tag{36}$$

The non-commutative plane was not considered so we fill this gap. It is related to the fuzzy sphere by a planar limit. We choose a suitable solution as

$$\mathcal{U} = \frac{1}{2}\left[c + R - \frac{R^2}{\sqrt{(y-R)^2 + 2u\bar{u}}}\right]. \tag{37}$$

In the limit $R \to \infty$ we obtain the plane solution $\mathcal{U} = (c - y)/2$.

Mirror multiplet (1,4,3)

The mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is described by a real superfield X satisfying

$$D^{(i}_{\alpha} D^{j)\alpha} X = 0 \quad \Rightarrow \quad D^{+}_{\alpha} D^{+\alpha} X = 0, \qquad D^{++} X = 0. \tag{38}$$

Solving them we obtain that

$$X = x - \theta^{-}_{\alpha}\psi^{i\alpha}u^{+}_{i} + \theta^{+}_{\alpha}\psi^{i\alpha}u^{-}_{i} + \theta^{-}_{(\alpha}\theta^{+}_{\beta)}A^{\alpha\beta} + i\theta^{-}_{\alpha}\theta^{+\alpha}\dot{x} + i\theta^{+\alpha}\theta^{+}_{\alpha}\theta^{-}_{\beta}\dot{\psi}^{i\beta}u^{-}_{i}\,, \tag{39}$$

where

$$\overline{(x)} = x, \qquad \overline{(\psi^{i\alpha})} = \psi_{i\alpha}\,, \qquad \overline{(A^{\alpha\beta})} = -\,A_{\alpha\beta}\,. \tag{40}$$

Supersymmetry transformations are

$$\delta x = \epsilon_{i\alpha}\psi^{i\alpha}, \qquad \delta\psi^{i\alpha} = \epsilon^{i}_{\beta}A^{\alpha\beta} + i\epsilon^{i\alpha}\dot{x}, \qquad \delta A^{\alpha\beta} = 2i\epsilon^{i(\alpha}\dot{\psi}^{\beta)}_{i}. \tag{41}$$

The kinetic Lagrangian for the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is constructed as

$$S_{\text{kin.}} = \int dt\,\mathcal{L}_{\text{kin.}} = \int d\zeta_{\text{H}}\, f\,(X)\,. \tag{42}$$

Coupling

The Lagrangian describing the interaction of both multiplets is constructed as

$$S_{\text{int.}} = \int dt\, \mathcal{L}_{\text{int.}} = \frac{\mu}{2} \int d\zeta_{\text{A}}^{--}\, h^{++}. \tag{43}$$

We guess the function h^{++} as

$$h^{++} = \theta^{+\alpha} V_{\alpha\beta} \left(D^{+\beta} X \right) + \frac{1}{3}\, \theta^{+\alpha} X \left(D^{+\beta} V_{\alpha\beta} \right) + \frac{1}{3}\, \theta^{-}_{\gamma} \theta^{+\gamma} \left(D^{+\alpha} V_{\alpha\beta} \right) \left(D^{+\beta} X \right). \tag{44}$$

Indeed the dependence on θ^{-}_{α} vanishes since h^{++} is analytic:

$$D^{+\gamma} h^{++} = 0, \qquad D^{++} h^{++} \neq 0. \tag{45}$$

One can directly check that the action is invariant:

$$\delta S_{\text{int.}} = \frac{\mu}{2} \int d\zeta_{\text{A}}^{--}\, \delta h^{++} = \frac{\mu}{2} \int d\zeta_{\text{A}}^{--}\, D^{++} \delta h = 0, \qquad D^{+\gamma} \delta h = 0. \tag{46}$$

The component Lagrangian reads

$$\mathcal{L}_{\text{int.}} = \frac{\mu}{2} \left(x\, C + A^{\alpha\beta} v_{\alpha\beta} - \psi^{i\alpha} \chi_{i\alpha} \right). \tag{47}$$

Total Lagrangian

The total Lagrangian is a sum of three Lagrangians:

$$\mathcal{L}_{\text{tot.}} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{WZ}} + \mathcal{L}_{\text{int.}}\,. \tag{48}$$

Excluding the fermionic fields $\chi^{i\alpha}$ by their equations of motion we obtain the total Lagrangian:

$$\mathcal{L}_{\text{tot.}} = \mathcal{L}_{\text{kin.}} + i\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta} + \frac{\mu}{2}\,A^{\alpha\beta}v_{\alpha\beta} - \frac{\mu^2\mathcal{R}^{\alpha\beta}\psi^i_\alpha\psi_{i\beta}}{4\mathcal{R}^{\gamma\delta}\mathcal{R}_{\gamma\delta}} + C\left(\frac{\mu\,x}{2} + \mathcal{U}\right). \tag{49}$$

The $\mathcal{N}=4$ supersymmetric coupling of the multiplets generates the constraint

$$h = \mathcal{U} + \frac{\mu\,x}{2} \approx 0, \tag{50}$$

which relates one degree of freedom of the spin variables $v^{\alpha\beta}$ to the dynamical bosonic field x.

Nahm equations

Poisson brackets of the constraints result in the same matrix:

$$\begin{vmatrix} \{\pi_{\alpha\beta}, \pi_{\gamma\delta}\}_{\rm PB} & \{\pi_{\alpha\beta}, \mathcal{U}\}_{\rm PB} \\ \{\mathcal{U}, \pi_{\gamma\delta}\}_{\rm PB} & 0 \end{vmatrix} = \begin{vmatrix} \{\pi_{\alpha\beta}, \pi_{\gamma\delta}\}_{\rm PB} & \{\pi_{\alpha\beta}, h\}_{\rm PB} \\ \{h, \pi_{\gamma\delta}\}_{\rm PB} & 0 \end{vmatrix}. \tag{51}$$

We obtain the following Poisson (Dirac) brackets:

$$\{x,p\} = 1, \qquad \{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i\left(\varepsilon_{\alpha\gamma}\mathcal{R}_{\beta\delta} + \varepsilon_{\beta\delta}\mathcal{R}_{\alpha\gamma}\right)}{2\,\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}}, \qquad \{p, v_{\alpha\beta}\} = \frac{\mu\,\mathcal{R}_{\alpha\beta}}{2\,\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}}. \tag{52}$$

One can see that

$$\{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i}{\mu}\left(\varepsilon_{\alpha\gamma}\{p, v_{\beta\delta}\} + \varepsilon_{\beta\delta}\{p, v_{\alpha\gamma}\}\right). \tag{53}$$

These are Nahm equations and they are written in the standard form as

$$\{p, v_c\} = \frac{1}{2}\,\varepsilon_{abc}\{v_a, v_b\}, \qquad v_{\alpha\gamma} \to v_a, \qquad a = 1,2,3. \tag{54}$$

We obtained a model equivalent to the model constructed in S. Fedoruk, E. Ivanov, O. Lechtenfeld, JHEP **1206** (2012) 147.

Non-commutative plane

Let us take as an example the non-commutative plane. The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{tot.}} \quad &= \quad \mathcal{L}_{\text{kin.}} + \frac{i}{2}\,\mu\left(u\dot{\bar{u}} - \dot{u}\bar{u}\right) + \mu\left(A^{12}y + \frac{1}{\sqrt{2}}\,A^{22}\bar{u} - \frac{1}{\sqrt{2}}\,A^{11}u\right) - \mu\,\psi^i_1\psi_{i2} \\ &\quad + \frac{\mu\,C}{2}\left(x - y + c\right). \end{aligned} \tag{55}$$

The coordinate y, perpendicular to the plane, is directly related to the dynamical component as

$$y = x + c. \tag{56}$$

Indeed, the Dirac brackets satisfy the Nahm equations:

$$\{u, \bar{u}\} = \frac{i}{\mu}\,, \qquad \{y, u\} = 0, \qquad \{y, \bar{u}\} = 0. \tag{57}$$

Coupling with chiral multiplet

We split the triplet $V^{\alpha\beta}$ into complex and real superfields as

$$V^{12} = -\,Y, \qquad V^{22} = -\sqrt{2}\,U, \qquad V^{11} = \sqrt{2}\,\bar{U}. \tag{58}$$

The constraints become

$$D^i\bar{U} = 0, \qquad \bar{D}_i U = 0, \qquad \sqrt{2}\,D_i Y = \bar{D}_i\bar{U}, \qquad \sqrt{2}\,\bar{D}_i Y = -\,D_i U. \tag{59}$$

where

$$D^i = D^{i1}, \qquad \bar{D}^i = D^{i2}, \qquad \theta_i := \theta_{i1}\,, \qquad \bar{\theta}^i := \theta^i_2\,. \tag{60}$$

Obviously the complex superfield U is chiral, so we can couple it in the chiral subspace $\{t_{\mathrm{L}}, \theta_i\}$ with the standard chiral superfield Z. The latter describes the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. The chiral superfields are

$$\begin{aligned} Z &= z + \sqrt{2}\,\theta_k\xi^k + \theta_k\theta^k B, \\ U &= u - \frac{1}{\sqrt{2}}\,\theta_k\chi^k_1 - \frac{1}{2\sqrt{2}}\,\theta_k\theta^k\,(C + 2i\dot{y})\,. \end{aligned} \tag{61}$$

Interaction term

The interaction term is given by the superpotential

$$S_{\text{int.}} = \mu \int dt_{\text{L}}\, d^2\theta\, \mathcal{F}\left(Z, U\right) + \mu \int dt_{\text{R}}\, d^2\bar{\theta}\, \bar{\mathcal{F}}\left(\bar{Z}, \bar{U}\right). \tag{62}$$

The total Lagrangian:

$$\mathcal{L}_{\text{tot.}} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{pot.}} + \mathcal{L}_{\text{WZ}} + \mathcal{L}_{\text{int.}}\,. \tag{63}$$

Bosonic Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\text{tot.}} \quad = \quad & \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{pot.}} - \frac{i\,\mu}{\sqrt{2}}\left(\partial_u \mathcal{F} - \partial_{\bar{u}}\bar{\mathcal{F}}\right)\dot{y} + i\dot{y}\,\mathcal{A}_y + \sqrt{2}\,i\left(\dot{\bar{u}}\,\mathcal{A}_{\bar{u}} - \dot{u}\,\mathcal{A}_u\right) \\ & + \mu\left(\bar{B}\,\partial_{\bar{z}}\bar{\mathcal{F}} + B\,\partial_z\mathcal{F}\right) + C\left[\mathcal{U} - \frac{\mu}{2\sqrt{2}}\left(\partial_u\mathcal{F} + \partial_{\bar{u}}\bar{\mathcal{F}}\right)\right]. \end{aligned} \tag{64}$$

The interaction term $\mathcal{L}_{\text{int.}}$ contains first-order time derivatives $\sim \dot{y}$, *i.e.* it can be formally called interacting WZ Lagrangian.

Constraints

The relevant constraints are

$$\begin{aligned}
\pi_u &= p_u + \sqrt{2}\, i\, \mathcal{A}_u \approx 0, \\
\pi_{\bar u} &= p_{\bar u} - \sqrt{2}\, i\, \mathcal{A}_{\bar u} \approx 0, \\
\pi_y &= p_y - i\, \mathcal{A}_y + \frac{i\mu}{\sqrt{2}} \left[\partial_u \mathcal{F}\,(z,u) - \partial_{\bar u} \bar{\mathcal{F}}\,(\bar z, \bar u)\right] \approx 0, \\
h &= \mathcal{U}\,(y,u,\bar u) - \frac{\mu}{2\sqrt{2}} \left[\partial_u \mathcal{F}\,(z,u) + \partial_{\bar u} \bar{\mathcal{F}}\,(\bar z, \bar u)\right] \approx 0.
\end{aligned} \tag{65}$$

Here the last constraint imposes a more complicated relation between the dynamical complex boson z and the semi-dynamical triplet $(y, u, \bar u)$.

Dirac brackets

$$
\begin{aligned}
&\{z,p_z\}=1, \qquad \{\bar z,p_{\bar z}\}=1, \qquad \{p_z,p_{\bar z}\}=-\frac{i\mu^2\,\partial_u\partial_z\mathcal{F}\,\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}\,\partial_y\mathcal{U}}{2\,(\partial\mathcal{U})^2}, \qquad \{u,\bar u\}=-\frac{i\,\partial_y\mathcal{U}}{2\,(\partial\mathcal{U})^2},\\
&\{y,u\}=-\frac{i}{2\,(\partial\mathcal{U})^2}\left(\partial_{\bar u}\mathcal{U}-\frac{\mu\,\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}}\right), \qquad \{y,\bar u\}=\frac{i}{2\,(\partial\mathcal{U})^2}\left(\partial_{u}\mathcal{U}-\frac{\mu\,\partial_{u}\partial_{u}\mathcal{F}}{2\sqrt{2}}\right),\\
&\{p_z,y\}=-\frac{\mu\,\partial_u\partial_z\mathcal{F}\,\partial_y\mathcal{U}}{2\sqrt{2}\,(\partial\mathcal{U})^2}, \qquad \{p_z,u\}=-\frac{\mu\,\partial_u\partial_z\mathcal{F}}{\sqrt{2}\,(\partial\mathcal{U})^2}\left(\partial_{\bar u}\mathcal{U}-\frac{\mu\,\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}}\right),\\
&\{p_{\bar z},y\}=-\frac{\mu\,\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}\,\partial_y\mathcal{U}}{2\sqrt{2}\,(\partial\mathcal{U})^2}, \qquad \{p_{\bar z},\bar u\}=-\frac{\mu\,\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}}{\sqrt{2}\,(\partial\mathcal{U})^2}\left(\partial_{u}\mathcal{U}-\frac{\mu\,\partial_{u}\partial_{u}\mathcal{F}}{2\sqrt{2}}\right),\\
&(\partial\mathcal{U})^2=\left[\partial_y\mathcal{U}\,\partial_y\mathcal{U}+2\left(\partial_{\bar u}\mathcal{U}-\frac{\mu\,\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}}\right)\left(\partial_{u}\mathcal{U}-\frac{\mu\,\partial_{u}\partial_{u}\mathcal{F}}{2\sqrt{2}}\right)\right].
\end{aligned} \tag{66}
$$

Dirac brackets

$$
\begin{aligned}
&\{z, p_z\} = 1, \qquad \{\bar{z}, p_{\bar{z}}\} = 1, \qquad \{p_z, p_{\bar{z}}\} = -\frac{i\mu^2\,\partial_u\partial_z\mathcal{F}\,\partial_{\bar{u}}\partial_{\bar{z}}\bar{\mathcal{F}}\,\partial_y\mathcal{U}}{2\,(\partial\mathcal{U})^2}, \qquad \{u, \bar{u}\} = -\frac{i\,\partial_y\mathcal{U}}{2\,(\partial\mathcal{U})^2},\\
&\{y, u\} = -\frac{i}{2\,(\partial\mathcal{U})^2}\left(\partial_{\bar{u}}\mathcal{U} - \frac{\mu\,\partial_{\bar{u}}\partial_{\bar{u}}\bar{\mathcal{F}}}{2\sqrt{2}}\right), \qquad \{y, \bar{u}\} = \frac{i}{2\,(\partial\mathcal{U})^2}\left(\partial_u\mathcal{U} - \frac{\mu\,\partial_u\partial_u\mathcal{F}}{2\sqrt{2}}\right),\\
&\{p_z, y\} = -\frac{\mu\,\partial_u\partial_z\mathcal{F}\,\partial_y\mathcal{U}}{2\sqrt{2}\,(\partial\mathcal{U})^2}, \qquad \{p_z, u\} = -\frac{\mu\,\partial_u\partial_z\mathcal{F}}{\sqrt{2}\,(\partial\mathcal{U})^2}\left(\partial_{\bar{u}}\mathcal{U} - \frac{\mu\,\partial_{\bar{u}}\partial_{\bar{u}}\bar{\mathcal{F}}}{2\sqrt{2}}\right),\\
&\{p_{\bar{z}}, y\} = -\frac{\mu\,\partial_{\bar{u}}\partial_{\bar{z}}\bar{\mathcal{F}}\,\partial_y\mathcal{U}}{2\sqrt{2}\,(\partial\mathcal{U})^2}, \qquad \{p_{\bar{z}}, \bar{u}\} = -\frac{\mu\,\partial_{\bar{u}}\partial_{\bar{z}}\bar{\mathcal{F}}}{\sqrt{2}\,(\partial\mathcal{U})^2}\left(\partial_u\mathcal{U} - \frac{\mu\,\partial_u\partial_u\mathcal{F}}{2\sqrt{2}}\right),\\
&(\partial\mathcal{U})^2 = \left[\partial_y\mathcal{U}\,\partial_y\mathcal{U} + 2\left(\partial_{\bar{u}}\mathcal{U} - \frac{\mu\,\partial_{\bar{u}}\partial_{\bar{u}}\bar{\mathcal{F}}}{2\sqrt{2}}\right)\left(\partial_u\mathcal{U} - \frac{\mu\,\partial_u\partial_u\mathcal{F}}{2\sqrt{2}}\right)\right].
\end{aligned}
\tag{66}
$$

Deformation to SU(2|1) supersymmetry

Ordinary and mirror $\mathcal{N}=4$ multiplets admit deformations to SU(2|1) supersymmetry (E. Ivanov, S. Sidorov, Class. Quant. Grav. **31** (2014) 0750; J. Phys. A **47** (2014) 292002):

$$\begin{aligned}
\left\{Q^i_\beta, Q^\alpha_j\right\} &= 2\delta^i_j\delta^\alpha_\beta\left(H - mF\right) - 2m\left(\sigma_3\right)^\alpha_\beta I^i_j\,,\\
\left[I^i_j, I^k_l\right] &= \delta^k_j I^i_l - \delta^i_l I^k_j\,,\\
\left[I^i_j, Q^{k\alpha}\right] &= \delta^k_j Q^{i\alpha} - \frac{1}{2}\,\delta^i_j Q^{k\alpha},\\
\left[F, Q^{i\alpha}\right] &= \frac{1}{2}\left(\sigma_3\right)^\alpha_\beta Q^{i\beta}.
\end{aligned} \tag{67}$$

In the limit $m=0$, models of the standard $\mathcal{N}=4$ supersymmetric mechanics are restored with H being a central charge generator.

Deformation to SU(2|1) supersymmetry

- SU(2|1) supersymmetry breaks the equivalence between ordinary and mirror multiplets, because the first $\mathrm{SU}(2)_{\mathrm{L}}$ group becomes a subgroup of SU(2|1) and the second group $\mathrm{SU}(2)_{\mathrm{R}}$ is broken.

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