Super-Schwarzian mechanics

Nikolay Kozyrev

BLTP JINR, Dubna, Russia

in collaboration with S.Krivonos

based on arXiv:2111.04643

Schwarzian derivative

The Schwarzian derivative, defined by the relation

$$\{t,\tau\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2, \quad \dot{t} = \partial_\tau t,$$

has wide range of applications in mathematics, such as complex analysis and differential equations, and in physics. In particular, Schwarzian appears in the action that describes the d=2 Jackiw-Teitelboim gravity at the boundary

$$S = -\frac{1}{2}\int d\tau\,\{t,\tau\}\,,$$

which is important to study holographic correspondence (with Sachdev-Ye-Kitaev model in this case). Interesting property of the Schwarzian action is that its equation of motion is proportional to the derivative of the Schwarzian:

$$\delta S = 0 \quad \Rightarrow \quad \partial_\tau\,\{t,\tau\} = 0.$$

In conformal field theory, Schwarzian emerges when conformal transformations of the energy-momentum tensor are studied,

$$T(z) = \left(\frac{d\tilde{z}}{dz}\right)^2 \tilde{T}(\tilde{z}) + \{\tilde{z},z\}\,.$$

Main property of Schwarzians

Studies in conformal field theory suggest that the structure of appearing object $\{\tilde{z}, z\}$ is completely fixed by the conformal symmetry, and it would be desirable to find simpler way to compute such quantities. Also, in superconformal field theories, analogous inhomogeneous terms appear in the transformation laws of current superfield $J^{\mathcal{N}}$, which generates $\mathcal{N}$-extended superconformal transformations, and they can be considered as supersymmetric extensions of the Schwarzian derivative. It would be desirable to find a way to compute them, too, and obtain the supersymmetric extension of the Schwarzian action.

In this talk, we apply the method of nonlinear realizations to compute Schwarzians and find their supersymmetric analogs, using the known property of Schwarzian derivative — its invariance with respect to $SL(2,\mathbb{R})$ transformations:

$$\{t', \tau\} = \{t, \tau\} \ \ \text{iff} \ \ t' = \frac{at+b}{ct+d},$$

with $SL(2,\mathbb{R})$ being a finite-dimensional subgroup of conformal group in $d=1$. This suggests that the method of nonlinear realizations should be applied to $SL(2,\mathbb{R})$ to obtain this derivative.

The $sl(2,\mathbb{R})$ algebra and Cartan forms

The $sl(2,\mathbb{R})$ commutation relations can be written as

$$\mathrm{i}\,[D,P]=P,\quad \mathrm{i}\,[D,K]=-K,\quad \mathrm{i}\,[K,P]=2D.$$

where hermitean generators P, D, K can be identified with time shifts, dilatations and conformal boosts, respectively. Let us parameterize the $SL(2,\mathbb{R})$ group element just as done in standard conformal mechanics.

$$g=e^{\mathrm{i}tP}e^{\mathrm{i}zK}e^{\mathrm{i}uD},$$

where t, z, u are, so far, independent parameters. The transformations of parameters, induced by left multiplication, $g'=g_0 g$, in infinitesimal form read

$$g_0=e^{\mathrm{i}\tilde{a}P}\,e^{\mathrm{i}\tilde{b}D}\,e^{\mathrm{i}\tilde{c}K}\ \Rightarrow\ \delta t=\tilde{a}+\tilde{b}t+\tilde{c}\,t^2,\quad \delta u=\frac{d}{dt}\delta t,\quad \delta z=\frac{1}{2}\frac{d}{dt}\delta u-\frac{d}{dt}\delta t\,z.$$

The Cartan forms, strictly invariant with respect to left multiplication $g'=g_0 g$, are $\Omega=g^{-1}dg$, or

$$\Omega=\mathrm{i}\omega_P P+\mathrm{i}\omega_D D+\mathrm{i}\omega_K K,\ \ \omega_P=e^{-u}dt,\quad \omega_D=du-2zdt,\quad \omega_K=e^{u}\left(dz+z^2dt\right).$$

Schwarzian as projection of the K-form

Then standard superconformal mechanics are considered, t is taken as time, and u and z are treated as functions of t. When, one can enforce covariant condition $\omega_D = 0$, which would express z in terms of derivative of u, $z = 1/2du/dt$, and use the remaining forms to make up the action of conformal mechanics

$$S_{cf} = -\int \omega_K - g^2 \int \omega_P = \int dt \left[\left(\frac{dx}{dt} \right)^2 - \frac{g^2}{x^2} \right], \;\; x = e^{u/2}.$$

Let us, however, follow different path. Let us consider t, u, z as functions of some inert parameter τ; then, using the invariance of ω_P, ω_D and ω_K, one can enforce conditions

$$\omega_P = d\tau, \;\; \omega_D = 0 \;\; \Rightarrow \;\; e^u = \dot{t}, \;\; z = \frac{1}{2} e^{-u} \dot{u} = \frac{\ddot{t}}{2\dot{t}^2}.$$

When the remaining form ω_K, remarkably, turns out to be proportional to the Schwarzian:

$$\omega_K = \frac{1}{2} \left[\ddot{u} - \frac{1}{2} \dot{u}^2 \right] d\tau = \frac{1}{2} \left[\frac{\dddot{t}}{\dot{t}} - \frac{3}{2} \left(\frac{\ddot{t}}{\dot{t}} \right)^2 \right] d\tau, \;\; S = -\int \omega_K.$$

How to construct supersymmetric Schwarzians

The constuction described above would be straightforward to generalize to supersymmetric case. Then we should deal with some finite subalgebra of the whole superconformal algebra, which extends three mentioned $sl(2)$ generators P, D, K with some supercharges Q^i, superconformal charges S^i and, possibly, internal symmetry generators J^{ij}. Constructing the nonlinear realization of the respective group, we should treat all the group parameters as superfields that depend on some inert superspace coordinates τ, θ^i. Then the conditions

$$\omega_P = \triangle\tau, \;\; \left(\omega_Q\right)^i = d\theta^i, \;\; \omega_D = 0,$$

where the forms $\triangle\tau$ and $d\theta^i$ being invariant with respect to $\mathcal{N}$-extended supersymmetry, should express the remaining $\left(\omega_S\right)^i$, ω_K, $\left(\omega_J\right)^{ij}$ in terms of super Schwarzians and their derivatives.

Let us show how this program works in the case of $\mathcal{N}$=2 supersymmetry.

The $su(1,1|1)$ superalgebra

We are going to reconstruct $\mathcal{N}$=2 Schwarzian, starting from $su(1,1|1)$ superconformal algebra: $su(1,1|1)$

$$
\begin{aligned}
&\mathrm{i}\,[D,P]=P, \quad \mathrm{i}\,[D,K]=-K, \quad \mathrm{i}\,[K,P]=2D,\\
&\left\{Q,\overline{Q}\right\}=2P, \quad \left\{S,\overline{S}\right\}=2K, \quad \left\{Q,\overline{S}\right\}=-2D+2J, \left\{\overline{Q},S\right\}=-2D-2J,\\
&\mathrm{i}\,[J,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[J,\overline{Q}\right]=-\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[J,S]=\frac{1}{2}S,\ \mathrm{i}\left[J,\overline{S}\right]=-\frac{1}{2}\overline{S},\\
&\mathrm{i}\,[D,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[D,\overline{Q}\right]=\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[D,S]=-\frac{1}{2}S,\ \mathrm{i}\left[D,\overline{S}\right]=-\frac{1}{2}\overline{S},\\
&\mathrm{i}\,[K,Q]=-S,\ \mathrm{i}\left[K,\overline{Q}\right]=-\overline{S}, \quad \mathrm{i}\,[P,S]=Q,\ \mathrm{i}\left[P,\overline{S}\right]=\overline{Q}.
\end{aligned}
$$

Here, P, D, K are Hermitean, $J^\dagger=-J$ and $Q^\dagger=\overline{Q}$, $S^\dagger=\overline{S}$. We define the group element, in analog to the bosonic case, just as done in superconformal mechanics:

$$
g=e^{\mathrm{i}tP}\,e^{\xi Q+\bar{\xi}\overline{Q}}\,e^{\psi S+\bar{\psi}\overline{S}}e^{\mathrm{i}zK}e^{\mathrm{i}uD}e^{\phi J}
$$

Parameters of the group are superfields that depend on superconformally inert coordinates of some superspace τ (even), θ, $\bar{\theta}$ (odd). The forms, constructed of τ, θ, $\bar{\theta}$ and invariant with respect to $\mathcal{N}$=2 supersymmetry, are

$$
\triangle\tau=d\tau+\mathrm{i}(d\bar{\theta}\theta+d\theta\bar{\theta}),\ \ d\theta\ \ d\bar{\theta},\ \ \delta\tau=\mathrm{i}\big(\epsilon\bar{\theta}+\bar{\epsilon}\theta\big),\ \ \delta\theta=\epsilon,\ \ \delta\bar{\theta}=\bar{\epsilon}.
$$

The constraints

Using "boundary" supersymmetry invariant forms, one can define covariant derivatives with respect to τ, θ, $\bar\theta$:

$$d = d\tau \frac{\partial}{\partial\tau} + d\theta \frac{\partial}{\partial\theta} + d\bar\theta \frac{\partial}{\partial\bar\theta} = \triangle\tau D_\tau + d\theta D + d\bar\theta \overline{D} \;\Rightarrow$$

$$D_\tau = \partial_\tau,\; D = \frac{\partial}{\partial\theta} - \mathrm{i}\bar\theta \frac{\partial}{\partial\tau},\; \overline{D} = \frac{\partial}{\partial\bar\theta} - \mathrm{i}\theta \frac{\partial}{\partial\tau}, \qquad \left\{D, \overline{D}\right\} = -2\mathrm{i}\partial_\tau .$$

Equipped with these instruments, one can calculate the Cartan forms

$$g^{-1}dg = \mathrm{i}\omega_P P + \omega_Q Q + \bar\omega_Q \overline{Q} + \mathrm{i}\omega_D D + \omega_J J + \omega_S S + \bar\omega_S \overline{S} + \mathrm{i}\omega_K K$$

and find consequences of conditions $\omega_P = \triangle\tau$, $\omega_Q = d\theta$, $\bar\omega_Q = d\bar\theta$:

$$\omega_P = e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}\left(d\bar\xi\xi + d\xi\bar\xi\right)\right) = \triangle\tau \quad \Rightarrow \quad \begin{cases} \dot t + \mathrm{i}\left(\dot{\bar\xi}\xi + \dot\xi\bar\xi\right) = e^u, \\ Dt + \mathrm{i}D\xi\,\bar\xi = 0, \\ \overline{D}t + \mathrm{i}\overline{D}\bar\xi\,\xi = 0, \end{cases}$$

$$\begin{cases} \omega_Q = e^{-\frac{1}{2}(u-\mathrm{i}\phi)}\left(d\xi + \psi\triangle t\right) = d\theta \\ \bar\omega_Q = e^{-\frac{1}{2}(u+\mathrm{i}\phi)}\left(d\bar\xi + \bar\psi\triangle t\right) = d\bar\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \dot\xi + e^u\psi = 0,\; \dot{\bar\xi} + e^u\bar\psi = 0 \\ D\xi = e^{\frac{1}{2}(u-\mathrm{i}\phi)},\; \overline{D}\bar\xi = e^{\frac{1}{2}(u+\mathrm{i}\phi)} \\ \overline{D}\xi = 0\; D\bar\xi = 0. \end{cases}$$

The $\mathcal{N}$=2 Schwarzian

The remaining condition $\omega_D = 0$, if others are taken into account, implies

$$\omega_D = du - 2z\triangle t - 2i(d\xi\bar{\psi} + d\bar{\xi}\psi) = 0 \quad \Rightarrow \quad \dot{u} - 2e^u z = 0.$$

Then, after some calculations, the remaining Cartan forms ω_J, ω_K , ω_S, $\bar{\omega}_S$,

$$\begin{aligned}
\omega_J &= d\phi - 2\psi\bar{\psi}\triangle t + 2(d\bar{\xi}\psi - d\xi\bar{\psi}),\\
\omega_K &= e^u \left(dz + z^2\triangle t - \mathrm{i}(\psi\, d\bar{\psi} + \bar{\psi}\, d\psi) + 2\mathrm{i}z\,(d\xi\,\bar{\psi} + d\bar{\xi}\psi)\right),\\
\omega_S &= e^{\frac{u}{2}+\mathrm{i}\frac{\phi}{2}} \left(d\psi - \mathrm{i}\psi\bar{\psi}d\xi + z\,(d\xi + \psi\triangle t)\right),\\
\bar{\omega}_S &= e^{\frac{u}{2}-i\frac{\phi}{2}} \left(d\bar{\psi} + \mathrm{i}\psi\bar{\psi}d\bar{\xi} + z\,(d\bar{\xi} + \bar{\psi}\triangle t)\right)
\end{aligned}$$

can be written in terms of just one superfield quantity $\mathcal{S}$ and its derivatives, with some projections vanishing:

$$\begin{aligned}
\omega_J &= \mathrm{i}\mathcal{S}\triangle\tau, \;\; \omega_K = -\frac{1}{2}d\theta D\mathcal{S} + \frac{1}{2}d\bar{\theta}\overline{D}\mathcal{S} + \frac{1}{4}\left(\mathrm{i}\left[D, \overline{D}\right]\mathcal{S} - \mathcal{S}^2\right)\triangle\tau,\\
\omega_S &= -\frac{1}{2}d\theta\,\mathcal{S} - \frac{\mathrm{i}}{2}\triangle\tau\,\overline{D}\mathcal{S}, \quad \bar{\omega}_S = \frac{1}{2}d\bar{\theta}\mathcal{S} + \frac{\mathrm{i}}{2}\triangle\tau\, D\mathcal{S},\\
\mathcal{S} &= \frac{D\dot{\xi}}{D\xi} - \frac{\overline{D}\dot{\bar{\xi}}}{\overline{D}\bar{\xi}} - 2\mathrm{i}\frac{\dot{\xi}\dot{\bar{\xi}}}{D\xi\overline{D}\bar{\xi}} = \mathcal{S}_{\mathcal{N}=2}, \text{ which is } \mathcal{N}\text{=2 Schwarzian.}
\end{aligned}$$

$su(1,1|1)$ Maurer-Cartan equations

The structure of the Cartan forms in $\mathcal{N}$=2 case raises the question why this happens and if there a simpler means to show that the forms should have such a structure. Indeed, it is known that Cartan form $\Omega = g^{-1}dg$ satisfies so called Maurer-Cartan equation. We prefer to write it down in form that incorporates two *commuting* differentials d_1, d_2

$$d_2\Omega(d_1) - d_1\Omega(d_2) = [\Omega(d_1), \Omega(d_2)]\,.$$

Substitution $\Omega(d_i) = g^{-1}d_i g$ turns this relation into identity. At the same time, if general expression for Ω

$$\Omega = \mathrm{i}\omega_P P + \omega_Q Q + \bar{\omega}_Q \overline{Q} + \mathrm{i}\omega_D D + \omega_J J + \omega_S S + \bar{\omega}_S \overline{S} + \mathrm{i}\omega_K K$$

is substituted, it would be possible to find the relations structure functions of the forms satisfy.

Let us employ conditions

$$\omega_P = \triangle\tau,\ \ \omega_Q = d\theta,\ \ \bar{\omega}_Q = d\bar{\theta},\ \ \omega_D = 0$$

at this level. All other forms should be written as

$$\omega_J = \mathrm{i}\triangle\tau\mathcal{S} + d\theta\,\Phi - d\bar{\theta}\overline{\Phi},\ \ \omega_K = \triangle\tau C + d\theta\Sigma - d\bar{\theta}\overline{\Sigma},$$
$$\omega_S = \triangle\tau\,\Psi + d\theta A + d\bar{\theta}B,\ \ \bar{\omega}_S = \triangle\tau\,\overline{\Psi} + d\theta\overline{B} + d\bar{\theta}\overline{A}.$$

These should be substituted into Maurer-Cartan equations.

Solution to Maurer-Cartan equations

Maurer-Cartan equation, which comes with P generator, reads

$$d_2\omega_{1P} - d_1\omega_{2P} = -\left(\omega_{1P}\omega_{2D} - \omega_{1D}\omega_{2P}\right) + 2\mathrm{i}\left(\omega_{1Q}\bar{\omega}_{2Q} + \bar{\omega}_{1Q}\omega_{2Q}\right).$$

Upon substitution $\omega_P = \triangle\tau$, $\omega_Q = d\theta$, $\bar{\omega}_Q = d\bar{\theta}$, $\omega_D = 0$ it is satisfied identically due to $\triangle\tau = d\tau + \mathrm{i}\left(d\theta\bar{\theta} + d\bar{\theta}\theta\right)$. $d\omega_Q$ equation, however, has nontrivial consequences:

$$d_2\omega_{1Q} - d_1\omega_{2Q} = \omega_{1P}\omega_{2S} - \omega_{2P}\omega_{1S} + \frac{1}{2}\left(\omega_{1D}\omega_{2Q} - \omega_{2D}\omega_{1Q}\right) - \frac{\mathrm{i}}{2}\left(\omega_{1J}\omega_{2Q} - \omega_{2J}\omega_{1Q}\right)$$

$$d_2d_1\theta - d_1d_2\theta = 0 = \left(\triangle_1\tau\, d_2\theta - \triangle_2\tau d_1\theta\right)\left(A + \frac{1}{2}\mathcal{S}\right) + \left(\triangle_1\tau\, d_2\bar{\theta} - \triangle_2\tau d_1\bar{\theta}\right)B +$$

$$+\mathrm{i}d_1\theta\, d_2\theta\Phi + \frac{\mathrm{i}}{2}\left(d_1\bar{\theta}\, d_2\theta - d_2\bar{\theta}\, d_1\theta\right)\overline{\Phi}.$$

Just one equation is strong enough to show that the form ω_S can not have a $d\bar{\theta}$ -projection, and $d\theta$ and $d\bar{\theta}$ projections of ω_J are absent. Also it relates $d\theta$ projection of ω_S and $\triangle\tau$ projection of ω_J: $A = -1/2\mathcal{S}$. With these results, one can obtain from $d\omega_J$ equation

$$d_2\omega_{1J} - d_1\omega_{2J} = -2\left(\omega_{1Q}\bar{\omega}_{2S} - \bar{\omega}_{1Q}\omega_{2S} - \omega_{1S}\bar{\omega}_{2Q} + \bar{\omega}_{1S}\omega_{2Q}\right)$$

that $\Psi = -\frac{\mathrm{i}}{2}\overline{D}\mathcal{S}$, $\overline{\Psi} = \frac{\mathrm{i}}{2}D\mathcal{S}$, and ω_J, ω_S, $\bar{\omega}_S$ can be written entirely in terms of $\mathcal{S}$.

Solution to Maurer-Cartan equations

Continuing down this road, one can check that $d\omega_D$ equation

$$d_2\omega_{1D} - d_1\omega_{2D} = -2\big(\omega_{1P}\omega_{2K} - \omega_{1K}\omega_{2P}\big) - 2\mathrm{i}\big(\omega_{1Q}\bar{\omega}_{2S} + \bar{\omega}_{1Q}\omega_{2S} + \omega_{1S}\bar{\omega}_{2Q} + \bar{\omega}_{1S}\omega_{2Q}\big)$$

determines portions of ω_K form, $\Sigma = -\frac{1}{2}D\mathcal{S}$, $\overline{\Sigma} = -\frac{1}{2}\overline{D}\mathcal{S}$, and the function C can be determined by analyzing ω_S equation

$$d_2\omega_{1S} - d_1\omega_{2S} = -\omega_{1K}\omega_{2Q} + \omega_{2K}\omega_{1Q} - \frac{1}{2}\big(\omega_{1D}\omega_{2S} - \omega_{2D}\omega_{1S}\big) - \frac{\mathrm{i}}{2}\big(\omega_{1J}\omega_{2S} - \omega_{2J}\omega_{1S}\big)$$

and one obtains the already known structure of Cartan forms

$$\begin{aligned}
\omega_J &= \mathrm{i}\mathcal{S}\triangle\tau, \;\; \omega_K = -\frac{1}{2}d\theta D\mathcal{S} + \frac{1}{2}d\overline{\theta}\overline{D}\mathcal{S} + \frac{1}{4}\left(\mathrm{i}\left[D,\overline{D}\right]\mathcal{S} - \mathcal{S}^2\right)\triangle\tau, \\
\omega_S &= -\frac{1}{2}d\theta\,\mathcal{S} - \frac{\mathrm{i}}{2}\triangle\tau\,\overline{D}\mathcal{S}, \quad \bar{\omega}_S = \frac{1}{2}d\bar{\theta}\mathcal{S} + \frac{\mathrm{i}}{2}\triangle\tau\,D\mathcal{S}.
\end{aligned}$$

It is now matter of straightforward calculation to check that $d\omega_K$ equation

$$d_2\omega_{1K} - d_1\omega_{2K} = \big(\omega_{1K}\omega_{2D} - \omega_{1D}\omega_{2K}\big) + 2\mathrm{i}\big(\omega_{1S}\bar{\omega}_{2S} + \bar{\omega}_{1S}\omega_{2S}\big)$$

is satisfied, leaving no constraints on $\mathcal{S}$. It could be found only by studying Cartan forms within given group parametrization.

$\mathcal{N}$=2 Schwarzian action

With $\mathcal{S} = \mathcal{S}_{\mathcal{N}=2}$ being the sole invariant appearing in the forms, it is natural to assume that the superfield Schwarzian action is given by the integral

$$S_{N2schw} = -\frac{\mathrm{i}}{2}\int d\tau\, d\theta\, d\bar{\theta}\mathcal{S}_{\mathcal{N}=2}.$$

Keeping in mind structure of the forms, one can write it as

$$S_{N2schw} = -\frac{1}{2}\int \omega_J \wedge \omega_Q \wedge \bar{\omega}_Q = \mathrm{i}\int \omega_P \wedge \omega_S \wedge \bar{\omega}_Q = -\mathrm{i}\int \omega_P \wedge \omega_Q \wedge \bar{\omega}_S.$$

As an integral over odd variables is defined as $\int d\tau d\theta d\bar{\theta} = \int d\tau D\overline{D}$, one can evaluate the component action with the help of expression for ω_K:

$$S_{N2schw} = -\frac{1}{2}\int d\tau \left[\frac{\partial_\tau^2 \left(\dot{t} + \mathrm{i}\dot{\xi}\bar{\xi} + \mathrm{i}\dot{\bar{\xi}}\xi\right)}{\dot{t} + \mathrm{i}\dot{\xi}\bar{\xi} + \mathrm{i}\dot{\bar{\xi}}\xi} - \frac{3}{2}\frac{\left(\partial_\tau\left(\dot{t} + \mathrm{i}\dot{\xi}\bar{\xi} + \mathrm{i}\dot{\bar{\xi}}\xi\right)\right)^2}{\left(\dot{t} + \mathrm{i}\dot{\xi}\bar{\xi} + \mathrm{i}\dot{\bar{\xi}}\xi\right)^2} + \right.$$
$$\left. + 2\mathrm{i}\frac{\ddot{\xi}\dot{\bar{\xi}} + \ddot{\bar{\xi}}\dot{\xi}}{\left(\dot{t} + \mathrm{i}\dot{\xi}\bar{\xi} + \mathrm{i}\dot{\bar{\xi}}\xi\right)^2} - \frac{1}{2}\dot{\phi}^2 - 2\frac{\dot{\phi}\,\dot{\xi}\,\dot{\bar{\xi}}}{\dot{t}}\right].$$

Here, we denote the superfields and their first components with the same letter.

$osp(3|2)$ superalgebra

The construction of $\mathcal{N}$=3 Schwarzian is based on $osp(3|2)$ superalgebra

$$\begin{aligned}
&\mathrm{i}\,[D,P]=P,\quad \mathrm{i}\,[D,K]=-K,\quad \mathrm{i}\,[K,P]=2D,\\
&\{Q_i,Q_j\}=2\delta_{ij}P,\quad \{S_i,S_j\}=2\delta_{ij}K,\quad \{Q_i,S_j\}=-2\delta_{ij}D-\epsilon_{ijk}J_k,\\
&\mathrm{i}\,[D,Q_i]=\frac{1}{2}Q_i,\ \mathrm{i}\,[D,S_i]=-\frac{1}{2}S_i,\quad \mathrm{i}\,[K,Q_i]=-S_i,\ \mathrm{i}\,[P,S_i]=Q_i,\\
&\mathrm{i}\,[J_i,Q_j]=\epsilon_{ijk}Q_k,\quad \mathrm{i}\,[J_i,S_j]=\epsilon_{ijk}S_k,\quad \mathrm{i}\,[J_i,J_j]=\epsilon_{ijk}J_k.
\end{aligned}$$

All generators here are Hermitean, $i,j\ldots=1,2,3$, and $\epsilon_{ijk}=\epsilon_{[ijk]}$, $\epsilon_{123}=1$. The group element can be parameterized just as before,

$$g=e^{\mathrm{i}tP}\,e^{\xi_i Q_i}\,e^{\psi_j S_j}e^{\mathrm{i}zK}e^{iuD}e^{\mathrm{i}\phi_i J_i},$$

where group parameters are superfields that depend on τ, θ_i coordinates of superspace. The τ, θ_i are completely inert with respect to group transformations $g'=g_0 g$ and transform with respect to “boundary” supersymmetry as

$$\delta\tau=\mathrm{i}\epsilon_i\theta_i,\ \ \delta\theta_i=\epsilon_i\ \ \Rightarrow\ \ \delta\triangle\tau=\delta\big(d\tau+\mathrm{i}d\theta_i\theta_i\big)=0,\ \ \delta d\theta_i=0.$$

Solution to $osp(3|2)$ Maurer-Cartan equation

The covariant derivatives with respect to τ, θ_i are

$$d = d\tau \frac{\partial}{\partial \tau} + d\theta_i \frac{\partial}{\partial \theta_i} = \triangle\tau D_\tau + d\theta_i D_i \;\Rightarrow$$

$$D_\tau = \partial_\tau,\; D_i = \frac{\partial}{\partial\theta_i} - \mathrm{i}\theta_i \frac{\partial}{\partial\tau}, \qquad \{D_i, D_j\} = -2\mathrm{i}\delta_{ij}\partial_\tau.$$

To obtain the $\mathcal{N}$=3 Schwarzian, the Cartan forms

$$\Omega = g^{-1}dg = \mathrm{i}\omega_P P + (\omega_Q)_i Q_i + \mathrm{i}\omega_D D + \mathrm{i}(\omega_J)_i J_i + (\omega_S)_i S_i + \mathrm{i}\omega_K K$$

should be subjected to conditions

$$\omega_P = \triangle\tau,\;\; (\omega_Q)_i = d\theta_i,\;\; \omega_D = 0.$$

Taking into account results in $\mathcal{N}$=2, it would be useful to employ Maurer-Cartan equations to find the general structure of the forms after applying the conditions above. Without writing equations explicitly, the result is

$$\begin{aligned} (\omega_J)_i &= \mathrm{i}\triangle\tau D_i\mathcal{S} + d\theta_i\,\mathcal{S},\;\; (\omega_S)_i = \triangle\tau\Big(\mathcal{S}D_i\mathcal{S} - \frac{1}{2}\epsilon_{ipq}D_pD_q\mathcal{S}\Big) + \mathrm{i}\epsilon_{ijk}d\theta_j D_k\mathcal{S}, \\ \omega_K &= \triangle\tau\Big(-\mathrm{i}\mathcal{S}\dot{\mathcal{S}} + \frac{1}{6}(\epsilon_{pqr}D_pD_qD_r\mathcal{S}) - D_k\mathcal{S}D_k\mathcal{S}\Big) + \mathrm{i}d\theta_i\Big(\mathcal{S}D_i\mathcal{S} - \frac{1}{2}\epsilon_{ipq}D_pD_q\mathcal{S}\Big), \end{aligned}$$

with $\mathcal{S}$ being obvious $\mathcal{N}$=3 Schwarzian candidate.

The irreducibility conditions

To find the Schwarzian explicitly, we still need to find explicitly the Cartan forms and irreducibility conditions of the multiplet. The forms read

$$\omega_P = e^{-u}\left(dt - \mathrm{i}\,\xi_i d\xi_i\right) \equiv e^{-u}\triangle t, \;\; \omega_D = du - 2z\triangle t - 2\mathrm{i}\, d\xi_i\psi_i,$$
$$\omega_K = e^{u}\left(dz + z^2\triangle t - \mathrm{i}\,\psi_i d\psi_i - 2\mathrm{i}\, z\psi_i d\xi_i\right),$$
$$\left(\omega_Q\right)_i = \left(\hat{\omega}_Q\right)_j M_{ij}, \;\; \left(\omega_S\right)_i = \left(\hat{\omega}_S\right)_j M_{ij}, \;\; \left(\omega_J\right)_i = \left(\hat{\omega}_J\right)_j M_{ij} + \frac{1}{2}\epsilon_{ijk} dM_{jm}M_{km},$$

where hatted forms are

$$(\hat{\omega}_Q)_i = e^{-\frac{u}{2}}\left(d\xi_i + \triangle t\psi_i\right), \;\; (\hat{\omega}_J)_i = -\mathrm{i}\epsilon_{ijk}\left(\psi_j d\xi_k + \frac{1}{2}\triangle t\psi_j\psi_k\right),$$
$$(\hat{\omega}_S)_i = e^{\frac{u}{2}}\left(d\psi_i - \mathrm{i}\psi_i\psi_j d\xi_j + z\left(d\xi_i + \triangle t\,\psi_i\right)\right).$$

The conditions $\omega_P = \triangle\tau$, $\left(\omega_Q\right)_i = d\theta_i$, $\omega_D = 0$ together imply that

$$\dot{t} + \mathrm{i}\dot{\xi}_i\,\xi_i = e^{u}, \;\; D_i\, t + \mathrm{i}D_i\xi_j\,\xi_j = 0, \;\; D_j\xi_k = e^{u/2}M_{jk}, \;\; \psi_k = -e^{-u}\dot{\xi}_k, \;\; z = \frac{1}{2}e^{-u}\dot{u}.$$

$\mathcal{N}=3$ Schwarzian action

Taking into account irreducibility conditions and their consequences, one can show that indeed

$$\left(\omega_J\right)_i = \ldots + d\theta_p\Big[-\mathrm{i}M_{ik}\epsilon_{klm}D_p\xi_l\psi_m + \frac{1}{2}\epsilon_{ijk}e^u D_pD_j\xi_m\ D_k\xi_m\Big] = \ldots + d\theta_i\mathcal{S}_{\mathcal{N}=3},$$

$$\mathcal{S}_{\mathcal{N}=3} = \frac{1}{6}e^{-u}\epsilon_{pqr}D_p\xi_n\,D_qD_r\xi_n = \frac{1}{2}\frac{\epsilon_{pqr}D_p\xi_n\,D_qD_r\xi_n}{D_k\xi_l\,D_k\xi_l}.$$

with $\mathcal{S}_{\mathcal{N}=3}$ being the known $\mathcal{N}$=3 Schwarzian. The Schwarzian action can also be constructed

$$\begin{aligned} S_{N3schw} &= -\frac{1}{6}\int d\tau\epsilon_{ijk}D_iD_jD_k\mathcal{S}_{\mathcal{N}=3} = -\frac{1}{6}\int \omega_P\wedge\left(\omega_Q\right)_i\wedge\left(\omega_Q\right)_j\wedge\left(\omega_J\right)_k\epsilon^{ijk} = \\ &= -\frac{1}{2}\int d\tau\Big[\frac{\partial_\tau^2\left(\dot t+\mathrm{i}\dot\xi_i\,\xi_i\right)}{\dot t+\mathrm{i}\dot\xi_i\,\xi_i} - \frac{3}{2}\left(\frac{\partial_\tau\left(\dot t+\mathrm{i}\dot\xi_i\,\xi_i\right)}{\dot t+\mathrm{i}\dot\xi_i\,\xi_i}\right)^2 + 2\mathrm{i}\frac{\dot\xi_i\ddot\xi_i}{\dot t+\mathrm{i}\dot\xi_i\,\xi_i} - \\ &\quad -2\mathrm{i}s\dot s - \mathrm{i}\frac{\dot M_{lp}M_{pm}\dot\xi_l\dot\xi_m}{\dot t+\mathrm{i}\dot\xi_i\,\xi_i} + \frac{1}{2}\dot M_{kl}\dot M_{kl}\Big]. \end{aligned}$$

Here, we denote the superfields and their first components with the same letter, and s is the first, independent, component of the Schwarzian.

The $su(1,1|2)$ superalgebra

Let us, finally, briefly describe the construction of the $\mathcal{N}$=4 supersymmetric Schwarzian. In $\mathcal{N}$=4 exists one-parametric family of superconformal algebras $D(2,1,a)$; we consider only $su(1,1|2)$, which corresponds to $a=-1$:

$$\big[D,P\big]=-\mathrm{i}P,\quad \big[D,K\big]=\mathrm{i}K,\quad \big[P,K\big]=2\mathrm{i}D,\quad \big\{Q_\alpha,\overline{Q}^\beta\big\}=2\delta_\alpha^\beta P,\quad \big\{S_\alpha,\overline{S}^\beta\big\}=2\delta_\alpha^\beta K,$$
$$\big\{Q_\alpha,\overline{S}^\beta\big\}=-2\delta_\alpha^\beta D-2T_\alpha{}^\beta,\quad \big\{\overline{Q}^\alpha,S_\beta\big\}=-2\delta_\beta^\alpha D+2T_\beta{}^\alpha,$$
$$\big[D,Q_\alpha\big]=-\tfrac{\mathrm{i}}{2}Q_\alpha,\quad \big[D,\overline{Q}^\alpha\big]=-\tfrac{\mathrm{i}}{2}\overline{Q}^\alpha,\quad \big[D,S_\alpha\big]=\tfrac{\mathrm{i}}{2}S_\alpha,\quad \big[D,\overline{S}^\alpha\big]=\tfrac{\mathrm{i}}{2}\overline{S}^\alpha,$$
$$\big[K,Q_\alpha\big]=\mathrm{i}S_\alpha,\quad \big[K,\overline{Q}^\alpha\big]=\mathrm{i}\overline{S}^\alpha,\quad \big[P,S_\alpha\big]=-\mathrm{i}Q_\alpha,\quad \big[P,\overline{S}^\alpha\big]=-\mathrm{i}\overline{Q}^\alpha.$$

The generators D,K,P commute with $su(2)$ generators $T_\alpha{}^\beta$, $T_\alpha{}^\alpha=0$; the commutators of $su(2)$ with themselves and fermionic generators read

$$\big[T_\alpha{}^\beta,T_\mu{}^\nu\big]=\mathrm{i}\big(\delta_\mu^\beta T_\alpha{}^\nu-\delta_\alpha^\nu T_\mu{}^\beta\big),$$
$$\big[T_\alpha{}^\beta,Q_\gamma\big]=\mathrm{i}\left(\delta_\gamma^\beta Q_\alpha-\frac{1}{2}\delta_\alpha^\beta Q_\gamma\right),\quad \big[T_\alpha{}^\beta,\overline{Q}^\gamma\big]=-\mathrm{i}\left(\delta_\alpha^\gamma\overline{Q}^\beta-\frac{1}{2}\delta_\alpha^\beta\overline{Q}^\gamma\right),$$
$$\big[T_\alpha{}^\beta,S_\gamma\big]=\mathrm{i}\left(\delta_\gamma^\beta S_\alpha-\frac{1}{2}\delta_\alpha^\beta S_\gamma\right),\quad \big[T_\alpha{}^\beta,\overline{S}^\gamma\big]=-\mathrm{i}\left(\delta_\alpha^\gamma\overline{S}^\beta-\frac{1}{2}\delta_\alpha^\beta\overline{S}^\gamma\right).$$

Here, indices $\alpha,\beta,\ldots=1,2$ can be raised and lowered with help of antisymmetric tensors $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma}=\delta_\alpha^\gamma$, $\epsilon_{12}=\epsilon^{21}=1$.

The $su(1,1|2)$ Cartan forms

The $SU(1,1|2)$ group element can be parameterized as

$$g = e^{\mathrm{i}tP} e^{\xi^\alpha Q_\alpha + \bar{\xi}_\alpha \overline{Q}^\alpha} e^{\psi^\alpha S_\alpha + \bar{\psi}_\alpha \overline{S}^\alpha} e^{\mathrm{i}zK} e^{\lambda_\beta{}^\alpha T_\alpha{}^\beta} e^{\mathrm{i}uD},$$

there parameters depend on inert “boundary” superspace coordinates τ, θ^α, $\bar{\theta}_\alpha$. The supersymmetry invariant forms and derivatives are

$$\triangle\tau = d\tau + \mathrm{i}d\theta^\alpha\,\bar{\theta}_\alpha + \mathrm{i}d\bar{\theta}_\alpha\,\theta^\alpha, \;\; \delta\tau = \mathrm{i}\big(\epsilon^\alpha\bar{\theta}_\alpha + \bar{\epsilon}_\alpha\,\theta^\alpha\big), \;\; \delta\theta^\alpha = \epsilon^\alpha, \;\; \delta\bar{\theta}_\alpha = \bar{\epsilon}_\alpha,$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}\bar{\theta}_\alpha\partial_\tau, \;\; \overline{D}^\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - \mathrm{i}\theta^\alpha\partial_\tau, \;\; \big\{D_\alpha, \overline{D}^\beta\big\} = -2\mathrm{i}\delta_\alpha^\beta\partial_\tau.$$

The left-invariant Cartan forms are defined in standard way:

$$g^{-1}dg = \mathrm{i}\omega_P P + \mathrm{i}\omega_K K + \mathrm{i}\omega_D D + \big(\omega_Q\big)^\alpha Q_\alpha + \big(\bar{\omega}_Q\big)_\alpha \overline{Q}^\alpha + \\ + \big(\omega_S\big)^\alpha S_\alpha + \big(\bar{\omega}_S\big)_\alpha \overline{S}^\alpha + \big(\omega_T\big)_\beta{}^\alpha T_\alpha{}^\beta.$$

Considering Maurer-Cartan equations with conditions $\omega_P = \triangle\tau$, $\big(\omega_Q\big)^\alpha = d\theta^\alpha$, $\big(\bar{\omega}_Q\big)_\alpha$, $\omega_D = 0$ applied, one can find that the $\big(\omega_T\big)_\alpha{}^\beta$ inevitably has structure,

$$\big(\omega_T\big)_\beta{}^\alpha = \mathcal{S}_\beta{}^\alpha \triangle\tau, \;\; \mathcal{S}_\alpha{}^\alpha = 0, \;\; D^{(\gamma}\mathcal{S}^{\alpha\beta)} = 0, \;\; \overline{D}^{(\gamma}\mathcal{S}^{\alpha\beta)} = 0,$$

with $\mathcal{S}_\alpha{}^\beta$ satisfying the $\mathcal{N}$=4, $d = 1$ vector multiplet conditions.

Solution to $su(1,1|2)$ Maurer-Cartan equations

The rest of the forms also can be written in terms of $\mathcal{S}_\alpha{}^\beta$:

$$(\omega_S)^\alpha = \frac{1}{3}\triangle\tau\overline{D}^\gamma\mathcal{S}_\gamma{}^\alpha - \mathrm{i}\mathcal{S}_\beta{}^\alpha d\theta^\beta, \quad (\bar{\omega}_S)_\alpha = -\frac{1}{3}\triangle\tau D_\gamma\mathcal{S}_\alpha{}^\gamma + \mathrm{i}\mathcal{S}_\alpha{}^\beta d\bar{\theta}_\beta,$$

$$\omega_K = \triangle\tau\left(\frac{1}{12}\left[D_\mu,\overline{D}_\nu\right]S^{\mu\nu} - \frac{1}{2}S_{\mu\nu}S^{\mu\nu}\right) - \frac{\mathrm{i}}{3}d\theta^\alpha D_\gamma\mathcal{S}_\alpha{}^\gamma + \frac{\mathrm{i}}{3}d\bar{\theta}_\alpha\overline{D}^\gamma\mathcal{S}_\gamma{}^\alpha.$$

To find $\mathcal{S}_\alpha{}^\beta$ explicitly, one again should calculate the Cartan forms and irreducibility conditions. The ω_P conditions read

$$\omega_P = e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}\left(d\xi^\alpha\bar{\xi}_\alpha + d\bar{\xi}_\alpha\xi^\alpha\right)\right) = \triangle\tau \;\Rightarrow\; \dot{t} + \mathrm{i}\left(\dot{\xi}^\alpha\bar{\xi}_\alpha + \dot{\bar{\xi}}_\alpha\xi^\alpha\right) = e^u,$$

$$D_\alpha t + \mathrm{i}\left(D_\alpha\xi^\beta\bar{\xi}_\beta + D_\alpha\bar{\xi}_\beta\xi^\beta\right) = 0, \quad \overline{D}^\alpha t + \mathrm{i}\left(\overline{D}^\alpha\xi^\beta\bar{\xi}_\beta + \overline{D}^\alpha\bar{\xi}_\beta\xi^\beta\right) = 0.$$

The $(\omega_Q)^\alpha$ and $(\bar{\omega}_Q)_\alpha$ conditions read

$$(\omega_Q)^\alpha = e^{-\frac{u}{2}}\left(e^{-\mathrm{i}\lambda}\right)_\rho{}^\alpha\left(d\xi^\rho + \triangle t\psi^\rho\right) = d\theta^\alpha \;\Rightarrow$$

$$\psi^\alpha = -e^{-u}\dot{\xi}^\alpha, \quad D_\beta\xi^\alpha = \left(e^{\mathrm{i}\lambda}\right)_\beta{}^\alpha e^{u/2}, \quad \overline{D}^\beta\xi^\alpha = 0,$$

$$(\bar{\omega}_Q)_\alpha = e^{-\frac{u}{2}}\left(e^{\mathrm{i}\lambda}\right)_\alpha{}^\rho\left(d\bar{\xi}_\rho + \triangle t\bar{\psi}_\rho\right) = d\bar{\theta}_\alpha \;\Rightarrow$$

$$\bar{\psi}_\alpha = -e^{-u}\dot{\bar{\xi}}_\alpha, \quad \overline{D}^\beta\bar{\xi}_\alpha = \left(e^{-\mathrm{i}\lambda}\right)_\alpha{}^\beta e^{u/2}, \quad D_\beta\bar{\xi}_\alpha = 0.$$

$\mathcal{N}$=4 superfield action

Using these relations, one transform the $(\omega_T)_\beta{}^\alpha$ form into

$$(\omega_T)_\beta{}^\alpha = \triangle\tau\left[-\mathrm{i}(e^{-\mathrm{i}\lambda})_\gamma{}^\alpha\partial_\tau(e^{\mathrm{i}\lambda})_\beta{}^\gamma - 2e^{-u}(e^{-\mathrm{i}\lambda})_\mu{}^\alpha(e^{\mathrm{i}\lambda})_\beta{}^\nu\,\dot{\xi}^\mu\,\dot{\bar{\xi}}_\nu + \delta^\alpha_\beta\dot{\xi}^\mu\,\dot{\bar{\xi}}_\mu e^{-u}\right] =$$
$$= \frac{1}{4}\triangle\tau\left([D_\beta,\overline{D}^\alpha] - \frac{1}{2}\delta^\alpha_\beta[D_\gamma,\overline{D}^\gamma]\right)\log\left(D_\mu\xi^\nu\,\overline{D}^\mu\bar{\xi}_\nu\right),$$

as expected. The Schwarzian action is

$$\begin{aligned}
S_{N4schw} &= -\frac{1}{12}\int d\tau[D_\mu,\overline{D}_\nu]S^{\mu\nu} = \frac{1}{6}\int d\tau d\theta^\alpha d\bar{\theta}_\beta\,\mathcal{S}_\alpha{}^\beta = \\
&= \frac{1}{6}\int(\omega_Q)^\alpha\wedge(\bar{\omega}_Q)_\beta\wedge(\omega_T)_\alpha{}^\beta = \frac{\mathrm{i}}{6}\int\omega_P\wedge(\omega_S)^\alpha\wedge(\bar{\omega}_Q)_\alpha = \\
&= -\frac{\mathrm{i}}{6}\int\omega_P\wedge(\omega_Q)^\alpha\wedge(\bar{\omega}_S)_\alpha.
\end{aligned}$$

$\mathcal{N}$=4 component action

The component action reads

$$S_{N4schw} = -\frac{1}{2}\int d\tau \left[\frac{\partial_\tau^2\left(\dot t + \mathrm{i}\dot\xi^\alpha\bar\xi_\alpha + \mathrm{i}\dot{\bar\xi}_\alpha\xi^\alpha\right)}{\dot t + \mathrm{i}\dot\xi^\alpha\bar\xi_\alpha + \mathrm{i}\dot{\bar\xi}_\alpha\xi^\alpha} - \frac{3}{2}\left(\frac{\partial_\tau\left(\dot t + \mathrm{i}\dot\xi^\alpha\bar\xi_\alpha + \mathrm{i}\dot{\bar\xi}_\alpha\xi^\alpha\right)}{\dot t + \mathrm{i}\dot\xi^\alpha\bar\xi_\alpha + \mathrm{i}\dot{\bar\xi}_\alpha\xi^\alpha}\right)^2 + \right.$$
$$+2\mathrm{i}\frac{\ddot\xi^\alpha\dot{\bar\xi}_\alpha + \ddot{\bar\xi}_\alpha\dot\xi^\alpha}{\dot t + \mathrm{i}\dot\xi^\beta\bar\xi_\beta + \mathrm{i}\dot{\bar\xi}_\beta\xi^\beta} - 2\left(\frac{\dot\xi^\mu\dot{\bar\xi}_\mu}{\dot t}\right)^2 + \left(e^{-\mathrm{i}\lambda}\right)_\rho{}^\beta\left(e^{-\mathrm{i}\lambda}\right)_\sigma{}^\alpha\partial_\tau\left(e^{\mathrm{i}\lambda}\right)_\rho{}^\alpha\,\partial_\tau\left(e^{\mathrm{i}\lambda}\right)_\beta{}^\sigma -$$
$$\left. -4\mathrm{i}\frac{\left(e^{-\mathrm{i}\lambda}\right)_\rho{}^\beta\partial_\tau\left(e^{\mathrm{i}\lambda}\right)_\beta{}^\sigma\dot\xi^\rho\dot{\bar\xi}_\sigma}{\dot t + \mathrm{i}\dot\xi^\alpha\bar\xi_\alpha + \mathrm{i}\dot{\bar\xi}_\alpha\xi^\alpha}\right].$$

Maxwell algebra Schwarzians

Finally, let us note that an analog of Schwarzian appears during study of flat space limit of Jackiw-Teitelboim gravity - Sachdev-Ye-Kitaev model correspondence. It is related to the Maxwell algebra,

$$\mathrm{i}\left[D,P\right]=P,\;\; \mathrm{i}\left[D,K\right]=K,\;\; \mathrm{i}\left[K,P\right]=2Z$$

with Z being central charge generator instead of dilatation generator. Repeating steps we did in the bosonic case, one can define the coset element, calculate the Cartan forms

$$g=e^{\mathrm{i}t(P+m^2K+qJ)}e^{\mathrm{i}zK}e^{\mathrm{i}uZ}e^{\mathrm{i}\phi D},\;\; g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_K K+\mathrm{i}\omega_D D+\mathrm{i}\omega_Z Z,$$
$$\omega_P=e^{-\phi}dt,\;\; \omega_K=e^{\phi}\left(dz-qzdt+m^2dt\right),\;\; \omega_D=d\phi\;\; \omega_Z=du-2zdt,$$

and, imposing conditions

$$\omega_P=d\tau,\;\; \omega_Z=0 \;\;\Rightarrow\;\; \dot{t}=e^{\phi},\;\; z=\frac{\dot{u}}{2\dot{t}},$$

one can obtain the related Schwarzian as projection of ω_K,

$$\omega_K=d\tau\,\dot{t}\left[\frac{1}{2}\left(\frac{\ddot{u}}{\dot{t}}-\frac{\dot{u}\ddot{t}}{\dot{t}^2}\right)+m^2\dot{t}-\frac{1}{2}q\dot{u}\right],$$

which is already known Schwarzian constructed for this purpose. It would be interesting to find supersymmetric extensions of this system also.

Conclusion

In this talk, we discussed the method of construction of bosonic and supersymmetric Schwarzians using the formalism of nonlinear realizations. It involves calculation of invariant Cartan forms of a given superconformal group and enforcement of conditions, which are almost universal for all groups

$$\omega_P = \triangle\tau, \quad \left(\omega_Q\right)^i = d\theta^i, \quad \omega_D = 0$$

Here, τ and θ^i are the coordinates of some “boundary” superspace, and $\triangle\tau$ and $d\theta^i$ being supersymmetry-invariant forms. In the cases of $\mathcal{N}=1,2,3,4$, the constraints on respective superconformal group forms express them in terms of supersymmetric Schwarzians. This can be proven in simplest way by study of Maurer-Cartan equations. It was also shown that supersymmetric Schwarzian actions are given by integrals of Schwarzians over appropriate superspaces.
This work can be extended further to more general superconformal groups, such as $D(2,1,a)$. Another interesting problem is to obtain non-relativistic version of Schwarzians.