The role of light polarization and related effects in Maxwell fish eye refractive profile

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Based on the following papers:

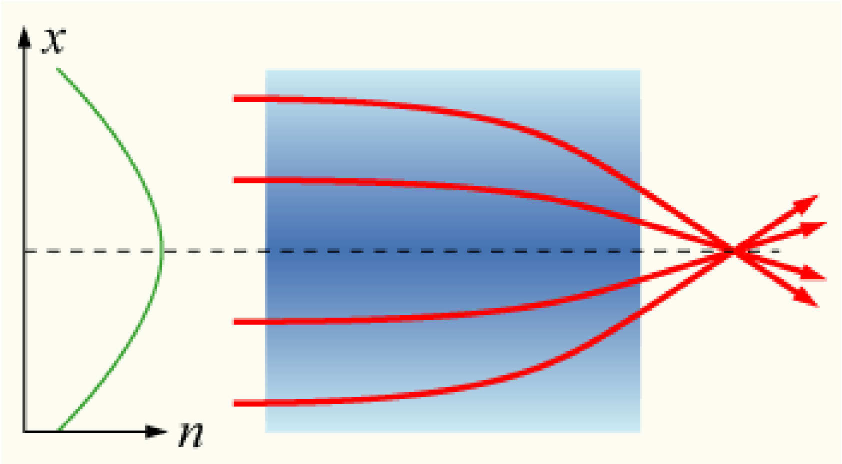
1. **M. D., Z. Gevorkian, & A. Nersessian (2021). Maxwell fish eye for polarized light. *Physical Review A, 104*(5), 053502.**

2. **Z. Gevorkian, M.D., & A. Nersessian (2020). Extended symmetries in geometrical optics. Physical Review A, 101(2), 023840.**

Structure

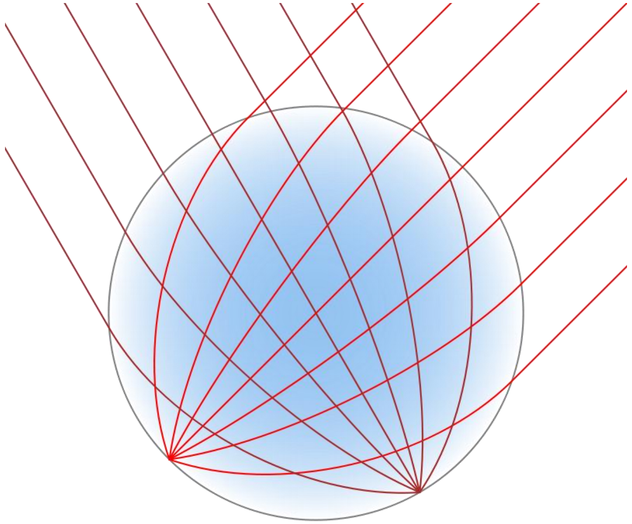
- Refractive index profiles, the “Maxwell fish eye”
- Hamiltonian formalism
- Construction scheme of optical Hamiltonians and refraction indices with similar symmetry algebras
- Coulomb-Fish eye “duality”
- Construction of MFE profile for polarized light
- Trajectories for polarized light
- Deformations of the trajectories

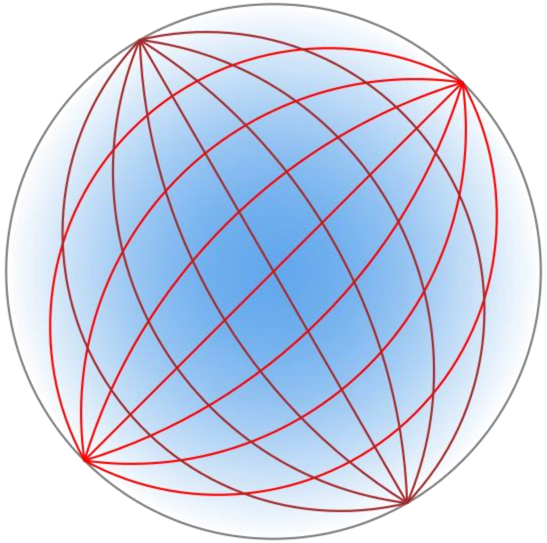
Refractive index profiles, the “Maxwell fish eye”.



- if the refractive index of a medium is not constant but has some sort of spatial dependency, the material is known as a gradient-index or GRIN medium
- the crystalline lens of the human eye is an example of a GRIN lens with a refractive index varying from about 1.406 in the inner core to approximately 1.386 at the less dense cortex
- Luneburg lens: $n(\mathbf{r}) = \sqrt{2-(\mathbf{r}/\mathrm{R})^2}$
- Maxwell fish eye lens (n_0 and R are constants):

$$n(\mathbf{r}) = \frac{\mathrm{n}_0}{1+(\boldsymbol{r}/R)^2}$$





Geometry

- action of a system on a three-dimensional curved space equipped with the "optical metrics" of Euclidean signature $d\tilde{l}^2 = n^2(\mathbf{r})d\mathbf{r}\cdot d\mathbf{r}$

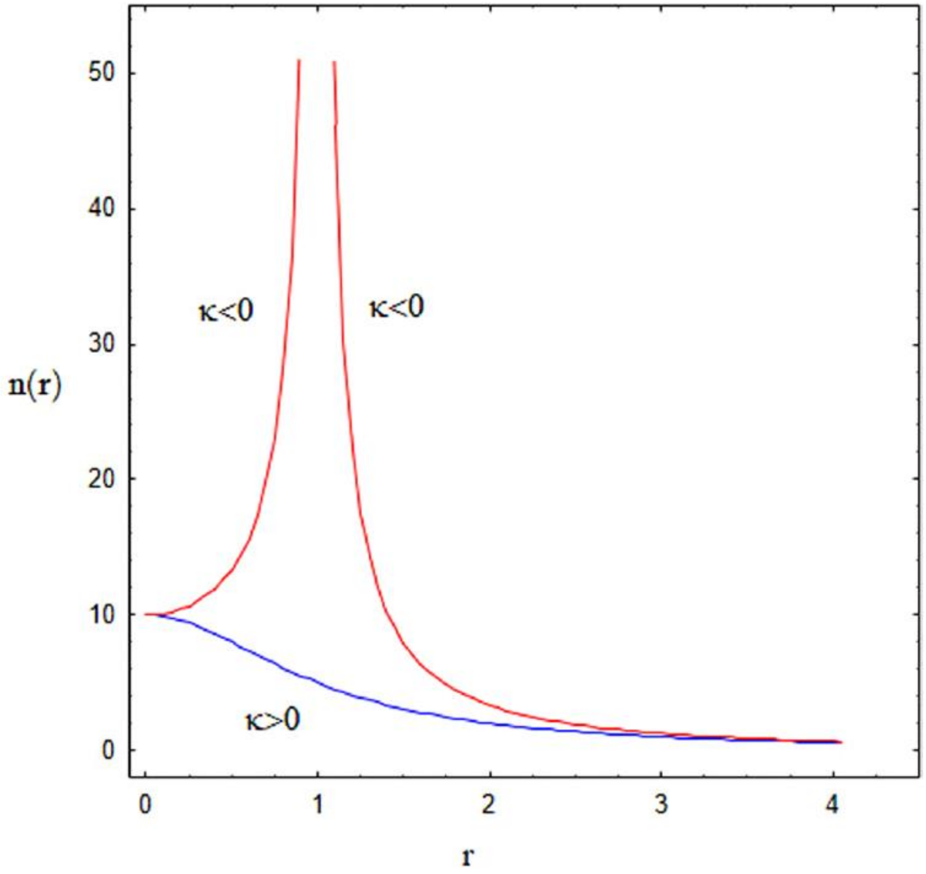
$$S_{Fermat} = \frac{1}{\lambda\!\!\!^{-}_0}\int d\tilde{l}, \qquad d\tilde{l} := n(\mathbf{r})|d\,\mathbf{r}/d\,\tau|d\tau$$

- MFE is defined by the metrics of (three-dimensional) sphere or pseudosphere (under pseudosphere we mean the upper (or lower) sheet of the two-sheet hyperboloid).

$$n(\mathbf{r}) = \frac{n_0}{|1+\kappa\mathbf{r}^2|}, \quad \kappa = \pm\frac{1}{4r_0^2}$$

- the symmetries of the system which describe the propagation of light in a particular medium are coming from the symmetries of the optical metrics of that particular medium

- SO(3) and SO(4)/SO(3.1)



Hamiltonian formalism

$$\mathcal{H}_0 = \alpha(\mathbf{p},\mathbf{r})\Phi = \alpha(\mathbf{p},\mathbf{r})\left(\frac{p^2}{n^2(\mathbf{r})} - {ƛ_0}^{-2}\right) \approx 0$$

$$\Phi \coloneqq \frac{\mathbf{p}^2}{n^2(\mathbf{r})} - {ƛ_0}^{-2} = 0.$$

$$\{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{x_i, x_j\} = 0,$$

$$\frac{df(r,p)}{d\tau} = \{f, \mathcal{H}_0\} = \{f, \alpha\}\Phi + \alpha\{f, \Phi\} \approx \alpha\{f, \Phi\}.$$

$$\alpha = \frac{n^2(\mathbf{r})}{p + ƛ_0^{-1} n(\mathbf{r})} \quad \Rightarrow \quad \mathcal{H}_{Opt} = p - ƛ_0^{-1} n(r)$$

$$\Downarrow$$

$$\frac{d\mathbf{p}}{dl} = ƛ_0^{-1} \nabla n(\mathbf{r}) \qquad \frac{d\boldsymbol{r}}{dl} = \frac{\boldsymbol{p}}{p}$$

Construction of optical Hamiltonians with similar symmetry algebras

- given the Hamiltonian $H = \frac{p^2}{2g(r)} + V(r)$ after fixing the energy surface *H* = *E*, (according to Maupertuis principle) we can relate the trajectories with the optical Hamiltonian $\mathcal{H}_0$ and with the refraction index $n(r) = \sqrt{2g(r)\big(E - V(r)\big)}$

- $\mathcal{H}_0$, $\mathcal{H}_{\mathrm{Opt}}$ as well as $n(r)$ inherit all the symmetries and constants of motion of the Hamiltonian
- canonical transformations preserve the symmetries of the Hamiltonians and their level surfaces

$$\Downarrow$$

we can to construct the physically non-equivalent optical Hamiltonians (and refraction indices) with the identical symmetry algebra

Coulomb-Fish eye "duality": Coulomb problem

$$H_{Coul} - E := \frac{p^2}{2} - \frac{\gamma}{r} - E = 0, \quad \Rightarrow \quad n_{Coul} = \lambda\!\!\!^{-}_0\sqrt{2(E + \gamma/r)}, \qquad \text{where} \quad \gamma > 0.$$

- **Integrals of motion:** $\mathbf{L} = r \times p, \quad \mathbf{A} = \mathbf{L} \times \mathbf{p} + \gamma \frac{\boldsymbol{r}}{r}$
- **Algebra formed:** $\{A_i, A_j\} = -2\varepsilon_{ijk} H_{Coul} L_k, \quad \{A_i, L_j\} = \varepsilon_{ijk} A_k, \quad \{L_i, L_j\} = \varepsilon_{ijk} L_k$

Coulomb-Fish eye "duality": Maxwell Fish eye

$$(\mathbf{p}, \mathbf{r}) \rightarrow (-\mathbf{r}, \mathbf{p})$$

$$r^2 - \frac{2\gamma}{p} - 2E = 0 \Rightarrow p - \frac{2\gamma}{r^2 - 2E} = 0 \quad \Rightarrow \quad n_{Mfe} = \frac{n_0}{|1+\kappa r^2|}, \text{where} \quad \kappa := -\frac{1}{2E}, \qquad \frac{n_0}{\lambda_0} := 2\epsilon\kappa$$

- **Integrals of motion:** $\mathbf{L} \rightarrow \mathbf{L}, \quad \mathbf{A} \rightarrow \frac{\mathbf{T}}{2\kappa}, \quad \mathbf{T} = (1 - \kappa \mathbf{r}^2)\mathrm{p} + 2\kappa(\mathrm{rp})\boldsymbol{r}$
- **Algebra formed:** $\{L_i, L_j\} = \varepsilon_{ijk} L_k, \quad \{T_i, L_j\} = \varepsilon_{ijk} T_k, \quad \{T_i, T_j\} = 4\kappa\varepsilon_{ijk} L_k$

Inclusion of polarization

Inclusion of polarization in terms of Hamiltonian formalism means to preserve the form of the Hamiltonian and replace the canonical Poisson brackets by the twisted ones

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = s\varepsilon_{ijk}F_k(p), \qquad \{p_i, p_j\} = 0,$$

F_k are the components of the Berry monopole $\mathbf{F}$:

$$\mathbf{F} := \frac{\partial}{\partial \mathrm{p}} \times \mathbf{A}(\mathrm{p}) = \frac{\mathbf{p}}{\mathrm{p}^3}$$

where

$\mathbf{A}$ is the vector-potential of the "Berry monopole" i.e. the potential of the magnetic (Dirac) monopole located at the origin of momentum space

Rotation generators in this space : $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + s\frac{\boldsymbol{p}}{p}$

Equations of motion: $\frac{d\boldsymbol{p}}{dl} = \lambda_0^{-1}\nabla n(\boldsymbol{r}), \qquad \frac{d\boldsymbol{r}}{dl} = \frac{\boldsymbol{p}}{p} - \frac{s}{\lambda_0}\boldsymbol{F} \times \nabla n(\boldsymbol{r})$

However, the above procedure, i.e. twisting the Poisson bracket with preservation of the Hamiltonian, violates the non-kinematical (hidden) symmetry of the system.

Construction of MFE profile for polarized light: MICZ-Kepler problem

- Coulomb problem in the presence of Dirac monopole (MICZ-Kepler problem)

twisted Poisson brackets

$$H_{MICZ} = \frac{p^2}{2} + \frac{s^2}{2r^2} - \frac{\gamma}{r} \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = s\varepsilon_{ijk}\frac{x_k}{r^3}, \qquad \{x_i, x_j\} = 0$$

- **Integrals of motion:** $\mathbf{J} = \mathbf{r} \times \mathbf{p} + s\frac{\mathbf{r}}{r} \quad \mathbf{A_s} = \mathbf{J} \times \mathbf{p} + \gamma\frac{\mathbf{r}}{r}$

- **Algebra:** same as in previous case with the replacement $(\mathbf{L}, \mathbf{A}) \rightarrow (\mathbf{J}, \mathbf{A_s})$

Construction of MFE profile for polarized light: Deformed MFE

$$(\mathbf{p}, \mathbf{r}) \to (-\mathbf{r}, \mathbf{p})$$

$$H_{MICZ} = \mathrm{E} \quad \Leftrightarrow \quad \mathrm{r}^2 + \frac{\mathrm{s}^2}{\mathrm{p}^2} - \frac{2\gamma}{\mathrm{p}} - 2\mathrm{E} = 0 \quad \Rightarrow$$

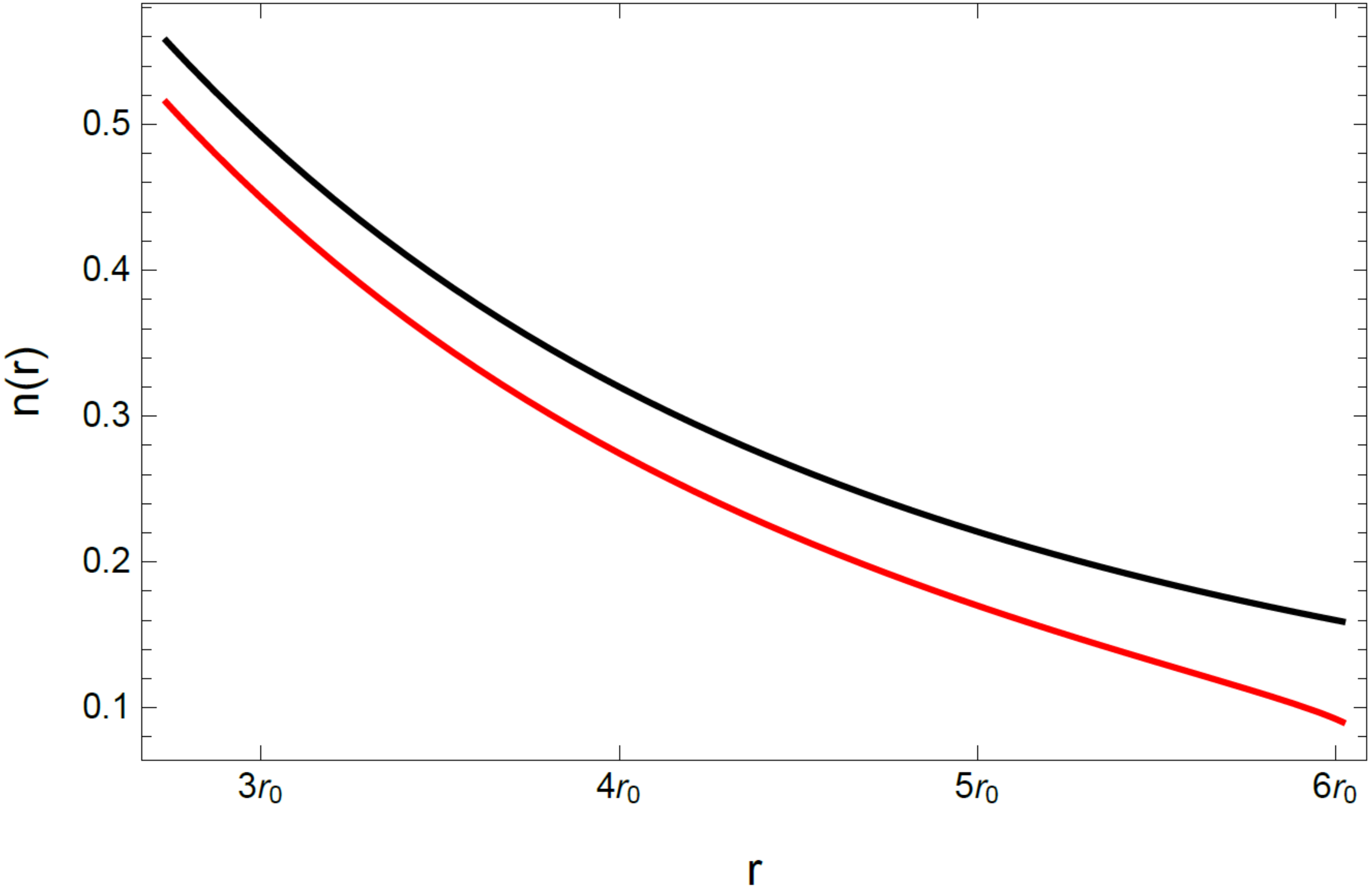
$$\mathrm{n}_{\mathrm{Mfe}}^{\mathrm{s}}(\mathbf{r}) = \frac{\mathrm{n}_{\mathrm{Mfe}}(\mathbf{r})}{2}\left(1 + \sqrt{1 - \frac{4\kappa \mathrm{s}^2 \lambda\!\!\!^{-}_0{}^2}{\mathrm{n}_0} \frac{1}{\mathrm{n}_{\mathrm{Mfe}}(\mathbf{r})}}\right),$$

where $\kappa := -\frac{1}{2\mathrm{E}}, \qquad \frac{\mathrm{n}_0}{\lambda\!\!\!^{-}_0} := 2\epsilon\kappa \quad , \epsilon = -\mathrm{sgn}(\mathrm{r}^2 + 1/\kappa)$

Construction of MFE profile for polarized light: Integrals of motion

- **Integrals of motion:** $\mathbf{J} \to \mathbf{r} \times \mathbf{p} + \mathrm{s}\frac{\mathbf{p}}{\mathrm{p}}$, $\mathbf{A_s} \to \frac{\boldsymbol{T_s}}{\kappa}$, $\boldsymbol{T_s} = \left(2 - \frac{\mathrm{n_0}}{\mathrm{n^s_{Mfe}}(\mathbf{r})}\right)\mathbf{p} + 2\kappa(rp)\mathbf{r} + \frac{2\kappa\mathrm{s}}{\mathrm{n^s_{Mfe}}(\mathrm{r})}\mathbf{J}$
- **Algebra:** same as the original MFE profile with the replacement $(\mathbf{L}, \mathbf{T}) \to (\mathbf{J}, \boldsymbol{T_s})$

MFE profile for polarized light



$$n^s_{Mfe}(\mathbf{r}) = \frac{n_{Mfe}(\mathbf{r})}{2}\left(1 + \sqrt{1 - \frac{4\kappa\lambda_0^2 s^2}{n_0}\frac{1}{n_{Mfe}(\mathbf{r})}}\right)$$

- for $\kappa > 0$ we get a restriction of rays in the finite domain: $r \leq \sqrt{\frac{n_0^2}{4s^2\lambda_0^2\kappa^2} - \frac{1}{\kappa}}$
- note that spin appears along with the factor $\kappa\lambda_0^2 = (\lambda_0/2r_0)^2$
- in order to stay within the bounds of geometrical optics approximation, this factor must be reasonably small.
- therefore, the influence of the spin will be far more notable within certain range of distance from the core of the fish eye.
- the latter happens when the condition $4\kappa s^2\lambda_0^2/n_0 \approx n_{Mfe}(r)$ holds
- at these distances the refraction index in the presence of spin can be much smaller compared to the refraction index with zero spin

Trajectories

$$\mathbf{r}\cdot\mathbf{J} = \mathrm{s}\frac{\mathbf{rp}}{\mathrm{p}},\quad \mathbf{r}\cdot\mathbf{T_s} = \frac{\mathrm{n}_0}{\lambda_0}\frac{\mathbf{rp}}{\mathrm{p}} \qquad \Rightarrow \qquad \mathbf{r}\cdot\left(\mathbf{J} - \frac{\mathrm{s}\lambda_0}{\mathrm{n}_0}\mathbf{T_s}\right) = 0 \quad \Rightarrow$$

ray trajectories are orthogonal to the axis $\boldsymbol{E_3} = \boldsymbol{J} - \frac{s\lambda_0}{n_0}\boldsymbol{T_s}$

$$\Downarrow$$

the trajectories belong to the plane spanned by the following vectors:

$$\mathbf{E_1} = \mathbf{T_s}\times\mathbf{J}, \qquad \mathbf{E_2} = \mathbf{E_3}\times\mathbf{E_1} = \left(\mathbf{J}^2 - \mathrm{s}^2\right)\left(\mathbf{T_s} - \frac{4\mathrm{s}\lambda_0\kappa}{\mathrm{n}_0}\mathbf{J}\right) \quad : \quad \mathbf{E_3}\cdot\mathbf{E_2} = \mathbf{E_3}\cdot\mathbf{E_1} = 0$$

Then, from the expression $\mathbf{J}\cdot(\mathbf{r}\times\mathbf{T_s})$ we immediately obtain the solution for the ray trajectories:

$$\mathbf{r}\cdot(\boldsymbol{T_s}\times\mathbf{J}) = (\mathrm{J}^2 - \mathrm{s}^2)\left(2 - \frac{\mathrm{n}_0}{\mathrm{n}_{\mathrm{mfe}}^{\mathrm{s}}}\right)$$

Trajectories: Polar coordinates

We introduce the following orthogonal frame

$$\mathbf{e}_{\mathrm{i}} = \frac{\boldsymbol{E}_{\boldsymbol{i}}}{|\mathrm{E}_{\mathrm{i}}|} \;:\quad \boldsymbol{e}_{\boldsymbol{i}} \cdot \boldsymbol{e}_{\boldsymbol{j}} = \delta_{ij}$$

$$|\boldsymbol{E}_{\boldsymbol{1}}|^2 = (\mathrm{J}^2 - \mathrm{s}^2)\left(\frac{\mathrm{n}_0^2}{\lambda_0^2} - 4\kappa \mathrm{J}^2\right), \qquad |\boldsymbol{E}_{\boldsymbol{3}}|^2 = (\mathrm{J}^2 - \mathrm{s}^2)\left(1 - \frac{4\mathrm{s}^2\lambda_0^2\kappa}{\mathrm{n}_0^2}\right), \qquad |\boldsymbol{E}_{\boldsymbol{2}}|^2 = |\boldsymbol{E}_{\boldsymbol{1}}|^2 |\boldsymbol{E}_{\boldsymbol{3}}|^2$$

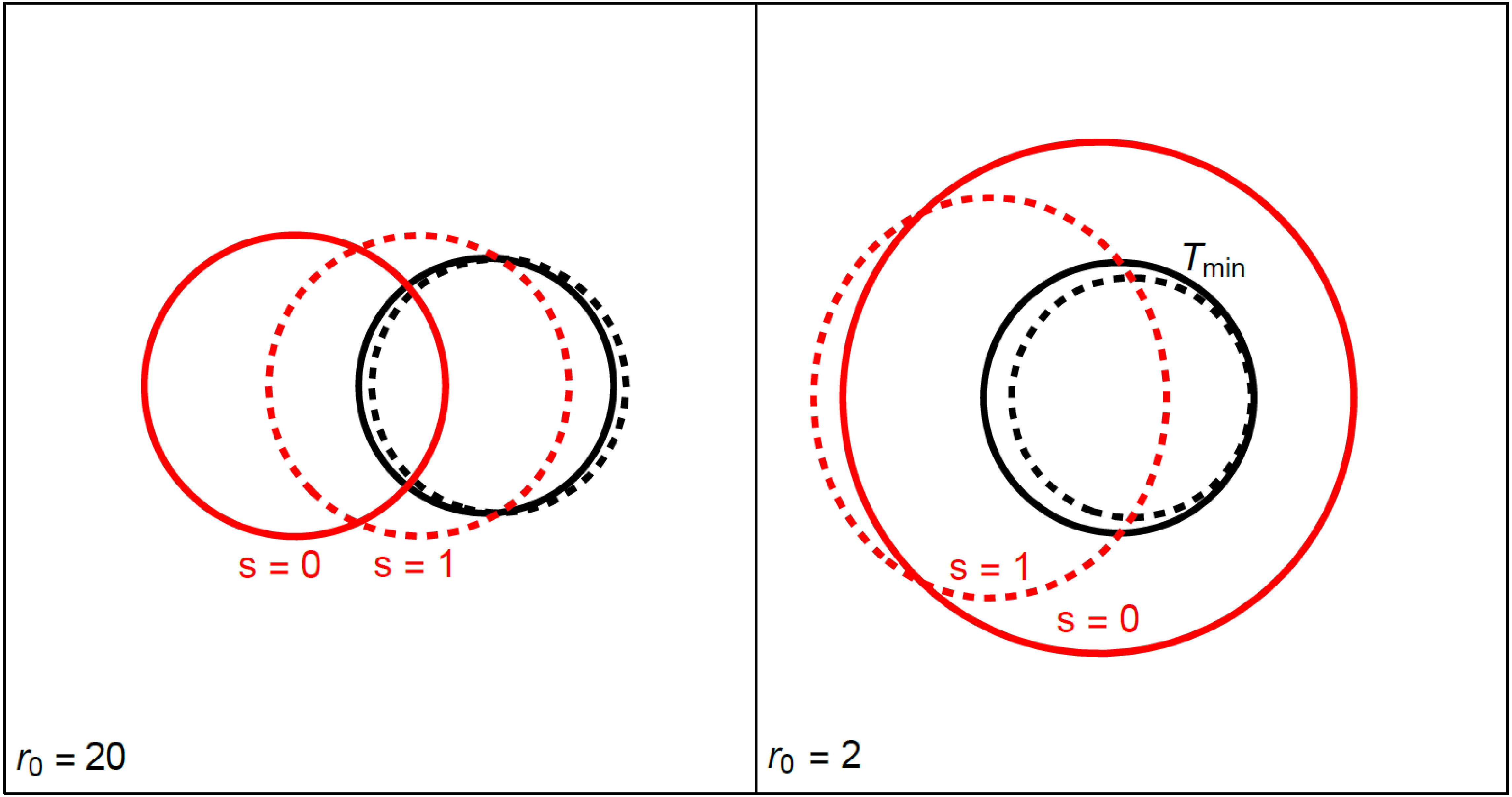
Decomposing ***r*** over this frame, we introduce the polar coordinates:

$\mathrm{r} = \underline{\mathrm{x}_1 \mathbf{e}_1} + \underline{\mathrm{x}_2 \mathbf{e}_2}$, $\quad \underline{\mathrm{x}_1} = \mathrm{r}\cos\varphi$, $\quad \underline{\mathrm{x}_2} = \mathrm{r}\sin\varphi$ and we get the trajectory equation

$$1 - |\kappa||a_s| r \cos\varphi = \frac{1 + \kappa r^2}{1 + \sqrt{1 - \frac{4\kappa s^2 \lambda_0^2}{n_0^2}(1 + \kappa r^2)}}$$

where $\quad |a_s|^2 \coloneqq R_s^2 - \frac{1}{\kappa}, \qquad R_s^2 \coloneqq \frac{n_0^2 - 4\kappa s^2 \lambda_0^2}{4\lambda_0^2 \kappa^2 (J^2 - s^2)}$

Deformations of the trajectories



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