RG flow between W_3 minimal models

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This talk is based on an yet unpublished paper titled: *RG flow between W_3 minimal models*

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The RG flow between neighboring minimal CFT models $A_2^{(p)}$ and $A_2^{(p-1)}$ with W_3 symmetry is explored. Although in perturbed theory dilatation current is no longer conserved it is still possible to get an exact operator expression for its divergence. Exploring this anomalous conservation law one can express the leading order anomalous dimensions of local fields in terms of structure constants of OPE in the original CFT. We generalize these line of argument for the case when a higher spin W current is present. We introduce the notion of anomalous W zero mode matrix which again can be expressed in terms of OPE coefficients of the original CFT.

Outline

1. Review on W_3 minimal CFTs and RG flow
   - Write the OPE and the W_3 algebra.
   - From Toda to minimal CFTs (map the Toda conformal (Virasoro) dimensions and w-weights to minimal ones.)
   - Identifying the IR fixed point
2. Matrix of anomalous dimensions
   - From dilatation current to the well known formula for the anomalous dimensions
   - Matrix of anomalous dimensions for the first two sets
3. Matrix of w anomalous weights
   - From W zero mode current to a formula for the matrix of w anomalous weights expressing it in terms of initial W_3 CFT weights and structure constants
   - Matrix of w anomalous weights for the second set. Diagonalization of this matrix provides an additional independent conformation that indeed $A_2^{(p)}$ flows to $A_2^{(p-1)}$.

W_3 CFTs

In any conformal field theory the energy-momentum tensor has two nonzero components: the holomorphic field $T(z)$ with conformal dimension $(2,0)$ and its anti-holomorphic counterpart $\bar{T}(\bar{z})$ with dimensions $(0,2)$. In W_3 CFTs one has in addition the currents $W(z)$ and $\bar{W}(\bar{z})$ with dimensions $(3,0)$ and $(0,3)$ respectively. These fields satisfy the OPE rules

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \cdots , \tag{1}$$

$$T(z)W(0) = \frac{3W(0)}{z^2} + \frac{W'(0)}{z} + \cdots , \tag{2}$$

$$W(z)W(0) = \frac{c/3}{z^6} + \frac{2T(0)}{z^4} + \frac{T'(0)}{z^3} + \frac{1}{z^2}\left(\frac{15c+66}{10(22+5c)}T''(0) + \frac{32}{22+5c}\Lambda(0)\right) + \frac{1}{z}\left(\frac{1}{15}T'''(0) + \frac{16}{22+5c}\Lambda'(0)\right) + \cdots .$$

Here $\Lambda(z)$ is a quasi primary field defined as: $\Lambda(z) =:TT:(z) - \frac{3}{10}T''(z)$, where :: is regularization by means of subtraction of all OPE singular terms. The second term is added to make $\Lambda(z)$ a quasiprimary field. Indeed the state created by this field: $\Lambda(0)|0\rangle = \left(L_{-2}^2 - \frac{3}{10}L_{-1}^2 L_{-2}\right)|0\rangle$ is a Virasoro quasi primary state i.e. $L_1\Lambda(0)|0\rangle = 0$.

We can expand these fields as Laurent series

$$T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \quad W(z) = \sum_{n=-\infty}^{+\infty} \frac{W_n}{z^{n+3}}, \quad \Lambda(z) = \sum_{n=-\infty}^{+\infty} \frac{\Lambda_n}{z^{n+4}}, \tag{3}$$

The OPE's (1), (2) and (3) are equivalent to the W_3 algebra relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{4}$$

$$[L_n, W_m] = (2n-m)\, W_{n+m}, \tag{5}$$

$$[W_n, W_m] = \alpha(n,m)L_{n+m} + \frac{16(n-m)}{22+5c}\Lambda_{n+m} + \frac{c}{360}(n^2-4)(n^2-1)n\delta_{n+m,0} \tag{6}$$

where

$$\alpha(n,m) = (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)$$

$$\Lambda_n = d_n L_n + \sum_{-\infty}^{+\infty} :L_m L_{n-m}: \tag{7}$$

here :: means normal ordering (i.e operators with smaller index come first) and

$$d_{2m} = \frac{1}{5}\left(1-m^2\right), \quad d_{2m-1} = \frac{1}{5}\left(1+m\right)\left(2-m\right). \tag{8}$$

The central charge of Virasoro algebra in A_2-Toda CFT conventionally is parametrized as

$$c = 2 + 12Q \cdot Q \,, \quad \text{where} \quad Q = \left(b + \frac{1}{b}\right)(\omega_1 + \omega_2)\,, \tag{9}$$

with b being the (dimensionless) Toda coupling and in what follows it would be convenient to represent the roots, weights and Cartan elements of A_2 as 3-component vectors with the usual Kronecker scalar product, subject to the condition that sum of components is zero. So

$$\omega_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \tag{10}$$

are the highest weights of two fundamental representations of $su(3)$. The weights of the fundamental representation are

$$h_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}; \quad h_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}; \quad h_3 = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix} \tag{11}$$

Conformal (Virasoro) dimensions and w-weights of the exponential fields V_α with charge α are given by

$$\Delta(\alpha) = \frac{\alpha \cdot (2\mathbf{Q} - \alpha)}{2}, \quad w(\alpha) = \frac{\sqrt{6}bi}{\sqrt{(3b^2+5)(5b^2+3)}} \prod_{i=1}^{3} ((\alpha - \mathbf{Q}) \cdot h_i) \;. \tag{12}$$

From Toda to minimal CFTs

To pass from the Toda theory to the minimal models, one specifies the parameter b as: $b = i\sqrt{\frac{p}{p+1}}$, where integers $p = 4, 5, 6, \cdots$ enumerate the infinite series of unitary models denoted as $A_2^{(p)}$. From (9) for the central charge we get: $c_p = 2 - \frac{24}{p(p+1)}$.

Furthermore, all primary fields of the minimal models are doubly-degenerated, a condition that is satisfied only for the following finite set of allowed charges

$$\alpha\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \frac{i\left(((n-1)(p+1)+(1-m)p)\omega_1 + ((n'-1)(p+1)+(1-m')p)\omega_2\right)}{\sqrt{p(p+1)}} \quad (13)$$

where, $n \geq 1, n' \geq 1, m \geq 1, m' \geq 1$ are integers subject to the additional constraints $n + n' \leq p - 1$, $m + m' \leq p$. In view of (12) the conformal and w dimensions are given explicitly by

$$\Delta\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \frac{((p+1)(m-m') - p(n-n'))^2 + 3((p+1)(m+m') - p(n+n'))^2 - 12}{12p(p+1)}$$

$$w\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \sqrt{6}((p+1)(n'-n) - p(m'-m)) \times \\ \times \frac{((p+1)(n+2n') - p(m+2m'))((p+1)(2n+n') - p(2m+m'))}{27p(p+1)\sqrt{(2p+5)(2p-3)}}$$

In what follows a special role is played by the field $\varphi(x)$ with the charge parameter $\alpha\left[\begin{smallmatrix}1 & 2\\ 1 & 2\end{smallmatrix}\right] = -b(\omega_1+\omega_2)$ and conformal dimension $\Delta(-b(\omega_1+\omega_2)) = \frac{p-2}{p+1} \equiv 1-\epsilon$, where $\epsilon = 3/(p+1)$ is introduced for later use. Notice also that the field φ is w-neutral: $w(-b(\omega_1+\omega_2)) = 0$. Consider the family generated from this field by multiple application of OPE. It is important that φ is the only member of this family (besides the identity operator) which is relevant. This fact allows one to construct a consistent perturbed CFT with a single coupling:

$$A = A_{CFT} + \lambda \int \varphi(x) d^2x \qquad (14)$$

At large values of p, $\epsilon << 1$ and the perturbing field is only slightly relevant and the conformal perturbation theory becomes applicable along a large portion of RG flow. The case of positive values of the coupling $\lambda > 0$ has been investigated in S. L. Lukyanov, V. Fateev [1] and it was shown that in infrared our initial theory $A_2^{(p)}$ flows to $A_2^{(p-1)}$.

[1] S. L. Lukyanov, V. Fateev , Additional Symmetries and Exactly Solv- able Models in Two Dimensional Conformal Field Theory: Physics Re- views, vol. 15, CRC Press, 1991.

In H.P. & R. Poghossian[2] one of our aims was to investigate this RG trajectory in more details.To find the matrices of anomalous dimensions we have derived several classes of OPE structure constants. Then we computed the matrices of anomalous dimensions for three RG invariant sets. The first set contains a single primary, the second one three primaries and the third includes six primaries and four level one secondary fields. We diagonalized the matrices of anomalous dimensions and found the explicit maps between UV and IR fields.

[2] RG flow between W_3 minimal models by perturbation and domain wall approaches.

Identifying the IR fixed point

For the diagonal structure constants we have:

$$C^{\varphi}_{\varphi,\varphi} \equiv C\begin{bmatrix} -b(\omega_1+\omega_2) \\ -b(\omega_1+\omega_2), -b(\omega_1+\omega_2)\end{bmatrix}$$
$$= \frac{2(4-5\rho)^2}{(3\rho-2)(4\rho-3)} \frac{\gamma^2\left(2-\frac{3\rho}{2}\right)\sqrt{\gamma(4-4\rho)\gamma(2-2\rho)}}{\gamma\left(1-\frac{\rho}{2}\right)\gamma\left(3-\frac{5\rho}{2}\right)\gamma(3-3\rho)}. \quad (15)$$

where $\rho = p/(p+1)$ also remind that $\epsilon = 3/(p+1)$. In the limit when $p \gg 1$ we get

$$C^{\varphi}_{\varphi,\varphi} = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\epsilon - \frac{4\sqrt{2}}{3}\epsilon^2 + O(\epsilon^3) \quad (16)$$
$$\beta(g) = \epsilon\, g - \frac{\pi}{2} C^{\varphi}_{\varphi,\varphi}\, g^2 + O(g^3) \quad (17)$$

Thus at $g = g_* = \frac{2\sqrt{2}\,\epsilon}{3\pi} + O\left(\epsilon^2\right)$ the beta-function vanishes and we get an infrared fixed point. The shift of the central charge is given by

$$c_p - c_* = 12\pi^2 \int_0^{g_*} \beta(g)dg = \frac{16}{9}\epsilon^3 + O(\epsilon^4). \quad (18)$$

On the other hand

$$c_p - c_{p-1} = \frac{48}{p(p^2-1)} = \frac{16}{9}\epsilon^3 + O(\epsilon^4), \quad (19)$$

It is well known that the slope of the beta function at a fixed point is directly related to the dimension of the perturbing field. In our case

$$\Delta_* = 1 - \left.\frac{d\beta}{dg}\right|_{g=g_*} = 1 + \epsilon + O(\epsilon^2) \tag{20}$$

As expected the perturbing slightly relevant field φ at UV becomes slightly irrelevant at IR. Remind that the W weight of φ is zero. It is possible to show that this weight should not get perturbative corrections at the IR point so that $w_0^{IR} = 0$. Examining (14) we see that the only primary field of $A_2^{(p-1)}$ with required properties is the field with charge

$$\alpha \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right] = -b^{-1}(\omega_1 + \omega_2) \tag{21}$$

indeed

$$\Delta \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]\Big|_{p\to p-1} = \frac{p+2}{p-1} = 1 + \epsilon + O(\epsilon^2)\,, \tag{22}$$

$$w \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]\Big|_{p\to p-1} = 0 \tag{23}$$

so that it can be identified with the perturbing field at the IR fixed point.

Matrix of anomalous dimensions

In perturbed theory $T = T_{zz}$ is no longer holomorphic, indeed

$$\bar{\partial}\langle T_{z,z}(z,\bar{z})\rangle_\lambda = \bar{\partial}\langle T(z)e^{-\int\lambda\varphi d^2x}\rangle_{CFT} =$$

$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int\bar{\partial}\langle T(z)\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_n =$$

$$\sum_{n=0}^{\infty}\sum_{i=1}^{n}\frac{(-\lambda)^n}{n!}\int\bar{\partial}\langle\varphi(x_1)\ldots\left(\frac{\Delta}{(z-x_i)^2}+\frac{1}{(z-x_i)}\frac{\partial}{\partial x_i}\right)\varphi(x_i)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_2$$

Where in the last step we have used the standard Ward identity. Now we can use

$$\bar{\partial}(z-x_i)^{-1} = \pi\delta^{(2)}(z-x_i), \quad \text{hence} \quad \bar{\partial}(z-x_i)^{-2} = -\pi\delta'(z-x_i) \quad (24)$$

and evaluate the integral. The result can be represented as

$$\bar{\partial}\langle T(z)e^{-\int\lambda\varphi d^2x}\rangle = -\pi\lambda(1-\Delta)\langle\varphi^{'}(z)e^{-\int\lambda\varphi d^2x}\rangle \quad (25)$$

Thus energy momentum conservation in the perturbed theory takes the form

$$\bar{\partial}T_{z,z}(z,\bar{z}) + \pi\lambda\epsilon\partial\varphi(z,\bar{z}) = 0 \quad (26)$$

Using this we immediately get

$$\bar{\partial}(zT) + \pi\epsilon\lambda\partial(z\varphi) = \pi\epsilon\lambda\varphi \quad (27)$$

The left hand side is the divergence of the unconserved current corresponding to dilatation. By definition the charge

$$\hat{Q}_T = \int_{\partial\Lambda} \left(zT\frac{dz}{2\pi i} - \pi\epsilon\lambda z\varphi\frac{d\bar{z}}{2\pi i} \right) \quad (28)$$

where Λ is a region of $\mathbb{R}^2$. Consider

$$\pi\epsilon\lambda\int_{\mathbb{R}^2}\varphi d\bar{z}dz = \pi\epsilon\lambda\int_{\Lambda}\varphi d\bar{z}dz + \pi\epsilon\lambda\int_{\mathbb{R}^2\setminus\Lambda}\varphi d\bar{z}dz = \pi\epsilon\lambda\int_{\Lambda}\varphi d\bar{z}dz + \quad (29)$$
$$+\int_{\mathbb{R}^2\setminus\Lambda}\left(\bar{\partial}(zT) + \pi\epsilon\lambda\partial(z\varphi)\right)d\bar{z}dz = \pi\epsilon\lambda\int_{\Lambda}\varphi d\bar{z}dz - \int_{\partial\Lambda}(zTdz - \pi\epsilon\lambda z\varphi dz)$$

the initial integral was independent of Λ thus the final expression is independent too.

So we can consider the ultraviolet limit taking Λ very small, notice also that due to the irrelevance of perturbation the effective coupling nearly vanishes, leading to the equality

$$-\int_{\partial\Lambda}\left(zT\phi_\beta(0)\frac{dz}{2\pi i}-\pi\epsilon\lambda z\varphi\phi_\beta(0)\frac{dz}{2\pi i}\right)+\pi\epsilon\lambda\int_\Lambda\varphi\phi_\beta(0)\frac{d\bar{z}dz}{2\pi i}= \\ =-\Delta_\beta\phi_\beta(0) \tag{30}$$

From (28)

$$\hat{Q}_T(\phi_\beta(0))=\Delta_\beta\phi_\beta(0)+\pi\epsilon\lambda\int_\Lambda\varphi\phi_\beta(0)\frac{d\bar{z}dz}{2\pi i} \tag{31}$$

which simply implies that

$$\hat{Q}_T(\phi_\beta)=\Delta_\beta\phi_\beta-\epsilon\lambda\partial_\lambda\phi_\beta \tag{32}$$

Let us change the basis of fields to such where the new fields satisfy $\langle\phi^\lambda_\alpha(1)\phi^\lambda_\beta(0)\rangle_\lambda = \delta_{\alpha\beta}$. It is not difficult to see that

$$\phi^\lambda_\beta = B_{\beta\gamma}\phi_\gamma, \quad \text{where} \quad B_{\beta\gamma} = \delta_{\beta\gamma} + \frac{\pi\lambda C^\beta_{\varphi,\gamma}}{\epsilon + \Delta_{\gamma\beta}} + O(\lambda^2) \tag{33}$$

From here by straightforward computation we get

$$\hat{Q}_T(\phi^\lambda_\beta) = \left(B_{\beta n}\Delta_n B^{-1}_{nm} - \epsilon\lambda B_{\beta n}\partial_\lambda B^{-1}_{nm}\right)\phi^\lambda_m - \epsilon\lambda\partial_\lambda\phi^\lambda_\beta \tag{34}$$

The matrix of anomalous dimensions is defined as

$$\hat{\Gamma} = B\Delta B^{-1} - \epsilon\lambda B(\partial_\lambda B^{-1}) \tag{35}$$

inserting here B from (33) we get

$$\Gamma_{\alpha\beta} = \Delta_\alpha\delta_{\alpha\beta} + \pi\lambda C^\alpha_{\varphi,\beta} + O(\lambda^2) \tag{36}$$

At this order λ can be replaced by renormalized coupling constant g since $g(\lambda) = \lambda + O(\lambda^2)$ (notice that the normalization scale was chosen to be 1). So, finally we get Zamolodchikov's formula

$$\Gamma_{\alpha\beta} = \Delta_\alpha\delta_{\alpha\beta} + \pi g C^\alpha_{\varphi,\beta} + O(g^2) \tag{37}$$

Matrix of anomalous dimensions for the first two sets

In this section we are going to derive the matrix of anomalous dimension Γ for the first two sets. The firs set contains only the field $\phi_n = \phi\left[\begin{smallmatrix} n & n \\ n' & n' \end{smallmatrix}\right]$. Its Virasoro dimension (14) for small ϵ is given by

$$\Delta\left[\begin{smallmatrix} n & n \\ n' & n' \end{smallmatrix}\right] = \frac{1}{27}\epsilon^2\left(n^2+nn'+n'^2-3\right) + \frac{1}{81}\epsilon^3\left(n^2+nn'+n'^2-3\right) + O\left(\epsilon^4\right) \quad (38)$$

To find Γ from (37) we have to know $C_{\varphi;\phi_n}^{\phi_n}$ which for small ϵ is

$$C_{\varphi;\phi_n}^{\phi_n} = \frac{\epsilon^2\left(n^2+nn'+n'^2-3\right)}{27\sqrt{2}} + O\left(\epsilon^3\right) \quad (39)$$

At fixed point for small ϵ we have $g = g^* = \frac{\left(2\sqrt{2}\epsilon\right)}{3\pi} + O(\epsilon^2)$ so

$$\Gamma_{\phi_n\phi_n} = \frac{1}{27}\epsilon^2\left(n^2+nn'+n'^2-3\right) + \frac{1}{27}\epsilon^3\left(n^2+nn'+n'^2-3\right) + O\left(\epsilon^4\right) \quad (40)$$

From (14) it is straightforward to see that this just coincides with the conformal dimension $\Delta\left[\begin{smallmatrix} n & n \\ n' & n' \end{smallmatrix}\right]$ for $A_2^{(p-1)}$. Thus we conclude that ϕ_n in the $A_2^{(p)}$ theory flows to ϕ_n in $A_2^{(p-1)}$.

The second set contains tree fields:

$$\phi_1 = \phi\left[\begin{smallmatrix} n & n+1 \\ n' & n' \end{smallmatrix}\right]\;; \quad \phi_2 = \phi\left[\begin{smallmatrix} n & n \\ n' & n'-1 \end{smallmatrix}\right]\;; \quad \phi_3 = \phi\left[\begin{smallmatrix} n & n-1 \\ n' & n'+1 \end{smallmatrix}\right] \tag{41}$$

Their conformal dimensions (14) are

$$\Delta\left[\begin{smallmatrix} n & n+1 \\ n' & n' \end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(-2n - n' - 1) + O\left(\epsilon^2\right) \tag{42}$$

$$\Delta\left[\begin{smallmatrix} n & n \\ n' & n'-1 \end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(n + 2n' - 1) + O\left(\epsilon^2\right) \tag{43}$$

$$\Delta\left[\begin{smallmatrix} n & n-1 \\ n' & n'+1 \end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(n - n' - 1) + O\left(\epsilon^2\right) \tag{44}$$

To derive the anomalous dimensions with (37) we have derived the appropriate structure constants. The diagonal structure constants are:

$$C^1_{\varphi;1} = \frac{2n(n+n'+3) + 3(n'+1)}{6\sqrt{2}n(n+n')} + O(\epsilon) \tag{45}$$

$$C^2_{\varphi;2} = \frac{n(2n'-3) + 2n'^2 - 6n' + 3}{6\sqrt{2}n'(n+n')} + O(\epsilon) \tag{46}$$

$$C^3_{\varphi;3} = \frac{2nn' + 3n - 3n' - 3}{6\sqrt{2}nn'} + O(\epsilon) \tag{47}$$

The off diagonal structure constants are

$$C^2_{\varphi;1} = \frac{1}{n+n'}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2-1\right)}{8nn'}} + O(\epsilon) \quad (48)$$

$$C^3_{\varphi;1} = \frac{1}{n}\sqrt{\frac{\left(n^2-1\right)\left(n'+1\right)(n+n'+1)}{8n'(n+n')}} + O(\epsilon) \quad (49)$$

$$C^3_{\varphi;2} = \frac{1}{n'}\sqrt{\frac{(n-1)\left(n'^2-1\right)\left(n+n'-1\right)}{8n(n+n')}} + O(\epsilon) \quad (50)$$

Thus we have all ingredients to derive the matrix of anomalous dimensions from

$$\Gamma_{\alpha\beta} = \Delta_\alpha \delta_{\alpha\beta} + \pi g C^\alpha_{\varphi,\beta} + O(g^2) \quad (51)$$

The result is

$$\Gamma_{11} = \frac{1}{3} - \frac{\epsilon}{9}\left(1 + 2n + n' - \frac{2n(n+n'+3) + 3(n'+1)}{n(n+n')}\right), \tag{52}$$

$$\Gamma_{22} = \frac{1}{3} - \frac{\epsilon}{9}\left(1 - n - 2n' - \frac{n(2n'-3) + 2(n'-3)n' + 3}{n'(n+n')}\right), \tag{53}$$

$$\Gamma_{33} = \frac{1}{3} + \frac{\epsilon}{9}\left(1 + n - n' - \frac{3(n'+1)}{nn'} + \frac{3}{n'}\right), \tag{54}$$

$$\Gamma_{12} = \Gamma_{21} = \frac{\epsilon}{3(n+n')}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2 - 1\right)}{nn'}}, \tag{55}$$

$$\Gamma_{13} = \Gamma_{31} = \frac{\epsilon}{3n}\sqrt{\frac{(n^2-1)\,(n'+1)(n+n'+1)}{n'(n+n')}}, \tag{56}$$

$$\Gamma_{23} = \Gamma_{32} = \frac{\epsilon}{3n'}\sqrt{\frac{(n-1)\,(n'^2-1)\,(n+n'-1)}{n(n+n')}} \tag{57}$$

The eigenvalues of this matrix are:

$$\Gamma_{diagonal} = \left\{\frac{1}{9}(3 + \epsilon(n - n' + 1)), \frac{1}{9}(3 - \epsilon(2n + n' - 1)), \frac{1}{9}(3 + \epsilon(n + 2n' + 1))\right\} \tag{58}$$

Using (14) we see that the fields $\phi\left[\begin{smallmatrix} n & n+1 \\ n' & n' \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n & n \\ n' & n'-1 \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n & n-1 \\ n' & n'+1 \end{smallmatrix}\right]$ in $A_2^{(p)}$ flow to the fields $\phi\left[\begin{smallmatrix} n+1 & n \\ n'-1 & n' \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n-1 & n \\ n' & n' \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n & n \\ n'+1 & n' \end{smallmatrix}\right]$ in $A_2^{(p-1)}$ as expected.

The matrix of w anomalous weights

In perturbed theory W is no longer holomorphic. Indeed consider

$$\bar{\partial}\langle W(z)e^{-\int \lambda\varphi d^2x}\rangle = \sum_{n=0}^{\infty}\sum_{i=1}^{n}\frac{(-\lambda)^n}{n!}$$

$$\int \bar{\partial}\langle\varphi(x_1)\ldots\left(\frac{1}{(z-x_i)^2}+\frac{2}{(\Delta+1)(z-x_i)}\frac{\partial}{\partial x_i}\right)W_{-1}\varphi(x_i)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_2$$

where the Ward identities together with $w_\varphi = 0$ and $W_{-2}\varphi(z_i) = \frac{2}{\Delta+1}\partial_{z_i}W_{-1}\varphi(z_i)$ were used. From (24) with simple manipulations we get

$$\bar{\partial}\langle W(z)e^{-\int \lambda\varphi d^2x}\rangle = -\lambda\pi\frac{1-\Delta}{1+\Delta}\langle(\partial W_{-1}\varphi(z))e^{-\int\lambda\varphi d^2x}\rangle \qquad (59)$$

so we obtain the conservation low

$$\bar{\partial}W_{zzz}(z,\bar{z}) + \pi\lambda\frac{1-\Delta}{\Delta+1}\partial W_{-1}\varphi(z,\bar{z}) = 0 \qquad (60)$$

Using this it is straightforward to see that

$$\bar{\partial}(z^2W) + \pi\lambda\frac{1-\Delta}{1+\Delta}\partial(z^2W_{-1}\varphi) = 2\pi\lambda\frac{1-\Delta}{1+\Delta}zW_{-1}\varphi \qquad (61)$$

The left hand side is the definition of the not anymore conserved current.

So the corresponding (unconserved) charge

$$\hat{Q}_W = \int_{\partial\Lambda} \left(z^2 W(z) \frac{dz}{2\pi i} - \pi\lambda \frac{1-\Delta}{1+\Delta} z^2 W_{-1}\varphi(z) \frac{d\bar{z}}{2\pi i} \right) \tag{62}$$

Similar to (29) by using (61) we can show that the following combination of integrals do not depend on the choice of Λ

$$-\int_{\partial\Lambda} \left(z^2 W(z)\phi_\beta(0) \frac{dz}{2\pi i} - \pi\lambda \frac{1-\Delta}{1+\Delta} z^2 W_{-1}\varphi(z)\phi_\beta(0) \frac{d\bar{z}}{2\pi i} \right) + \tag{63}$$

$$+2\pi\lambda \frac{1-\Delta}{1+\Delta} \int_{\Lambda} z W_{-1}\varphi(z)\phi_\beta(0) \frac{dz d\bar{z}}{2\pi i} = -w_\beta \phi_\beta(0)$$

where we have used the fact that for small Λ only the first term in the first integral contributes. But again the initial right hand side of this equation is independent of Λ so this equality holds for arbitrary Λ. Finally we obtain

$$\hat{Q}_W(\phi_\beta) = w_\beta \phi_\beta(0) + 2\pi\lambda \frac{\epsilon}{2-\epsilon} \int_{\Lambda} z W_{-1}\varphi(z)\phi_\beta(0) \frac{dz d\bar{z}}{2\pi i} \tag{64}$$

From general arguments for the anomalous w-weight matrix up to $O(g^2)$ we may write

$$\mathfrak{W}_{\alpha\beta} = w_\beta \delta_{\alpha\beta} + \pi g b_{\alpha\beta} C^{\alpha}_{\varphi;\beta} + O(g^2) \tag{65}$$

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with some unknown coefficients $b_{\alpha\beta}$. We do not know how to derive the coefficients $b_{\alpha\beta}$ in a rigorous manner. Nevertheless for the case of second set of fields already discussed we can say more. For self consistency at IR fixed $g = g_*$ Γ and $\mathfrak{W}$ should commute to be simultaneously diagonalizable. It is easily seen that this is possible only if $b_{\alpha\beta} = \frac{1}{\sqrt{6}}$ for $\alpha, \beta = 1, 2, 3$. So that for the matrix elements of $\mathfrak{W}$ for the second set we got

$$\mathfrak{W}_{11} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon}{9\sqrt{6}}\left(\frac{6}{n+n'} + \frac{3(n'+1)}{n(n+n')} - 2n - n' + 1\right), \tag{67}$$

$$\mathfrak{W}_{22} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon\left(n^2 n' + n\left(3n'^2 + n' - 3\right) + n'\left(2n'^2 + n' - 6\right) + 3\right)}{9\sqrt{6}n'(n+n')}, \tag{68}$$

$$\mathfrak{W}_{33} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon}{9\sqrt{6}nn'}(n(n'(n-n'+1)+3) - 3(n'+1)), \tag{69}$$

$$\mathfrak{W}_{12} = \mathfrak{W}_{21} = \frac{\epsilon}{3\sqrt{6}(n+n')}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2-1\right)}{nn'}}, \tag{70}$$

$$\mathfrak{W}_{13} = \mathfrak{W}_{31} = \frac{\epsilon}{3\sqrt{6}n}\sqrt{\frac{\left(n^2-1\right)(n'+1)(n+n'+1)}{n'(n+n')}}, \tag{71}$$

$$\mathfrak{W}_{23} = \mathfrak{W}_{32} = \frac{\epsilon}{3\sqrt{6}n'}\sqrt{\frac{(n-1)\left(n'^2-1\right)(n+n'-1)}{n(n+n')}} \tag{72}$$

The eigenvalues of it are

$$\mathfrak{W}_{diag} = \left\{\frac{2+\epsilon(n-n'+1)}{9\sqrt{6}}, \frac{2-\epsilon(2n+n'-1)}{9\sqrt{6}}, \frac{2+\epsilon(n+2n'+1)}{9\sqrt{6}}\right\} \tag{73}$$

These coincides with the w dimension of $\phi\begin{bmatrix} n+1 & n \\ n'-1 & n' \end{bmatrix}$ $\phi\begin{bmatrix} n-1 & n \\ n' & n' \end{bmatrix}$ $\phi\begin{bmatrix} n & n \\ n'+1 & n' \end{bmatrix}$ in $A_2^{(p-1)}$ as expected.

SUMMARY

- By studying the unconserved charge corresponding to dilatation we consistently arrived to well known expression for the matrix of anomalous dimensions i.e.

$$\Gamma_{\alpha\beta} = \Delta_\alpha \delta_{\alpha\beta} + \pi g C^{\alpha}_{\varphi,\beta} + O(g^2) \tag{74}$$

  Using this formula we derived Γ for the first two sets (in the paper also for the third set). And found to which fields the fields in these sets flow in the IR.

- Analogously we studied the unconserved charge corresponding to the W zero mode. We concluded that one can write an expression for the matrix of anomalous w-weights of the following form

$$\mathfrak{W}_{\alpha\beta} = w_\beta \delta_{\alpha\beta} + \pi g b_{\alpha\beta} C^{\alpha}_{\varphi;\beta} + O(g^2) \tag{75}$$

  with some unknown coefficients $b_{\alpha\beta}$. For the first two sets we determined this coefficient to be $b_{\alpha\beta} = \frac{1}{\sqrt{6}}$. We hope that our knowledge of the Γ matrix for the third set will help us to determine $b_{\alpha\beta}$ in general.