Integrable Systems

Near Horizon Extremal Myers-Perry Black Holes

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Introduction

What is Myers-Perry Black Hole?

Myers-Perry (MP) Black Hole is the higher dimensional generalization of the rotating Kerr BH.

- In $d = 4$ dimensions the MP BH reduces to the Kerr BH.
- Setting all rotation parameters a_i to 0 will reduce the d dimensional MP BH to d dimensional Schwarzschild (non-rotating) BH. Now, also setting $M = 0$ yields the flat space metric.
- The form of MP metrics differs slightly for odd and even dimensions.

What is Extremal MP Black Hole?

- Event horizon of Kerr BH is described by

$$r_H = M + \sqrt{M^2 - a^2}$$

- It follows that BHs with $a > M$ are not physical.
- r_H is real only for $a \leq M$. Black holes with $a = M$ are called **extremal** (BHs with biggest possible angular momentum $J = M^2$ for given BH mass)
- This discussion can be generalized for MP black hole

What is Near Horizon Limit?

Near Horizon Limit (NHL) is a vacuum solution of Einstein equations, which describe the space-time near the event horizon of extremal Kerr BH.

- Naturally, one can assume that NHL can be obtained by redefining the radial coordinate r in the metric of the extremal Kerr BH

$$r \longrightarrow r_H + \epsilon r_H r \qquad \text{with } \epsilon \longrightarrow 0$$

- But this redefinition gives rise to a degenerate metric. The problem can be resolved by taking additional limits

$$t \longrightarrow \frac{\alpha t}{\epsilon}, \qquad \phi_i \longrightarrow \phi_i + \frac{\beta_i t}{\epsilon}$$

Near Horizon limit of an Extremal Myers-Perry Black Hole

NHEMP

NHEMP geometry slightly differs in odd and even dimensions. For that reason we introduce a unified description for arbitrary dimensions

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(x,\sigma) x_i x_j D\varphi^i D\varphi^j$$

where

$$N_\sigma = N + \sigma, \qquad \sigma = \begin{cases} 0 & \text{when} \quad D = 2N+1 \\ 1 & \text{when} \quad D = 2N+2 \end{cases}$$

NHEMP

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(\sigma) x_i x_j D\varphi^i D\varphi^j$$

- Latitudinal coordinates x_I and rotation parameters m_I are restricted:

$$\sum_{I=1}^{N_\sigma} \frac{x_I^2}{m_I} = 1, \qquad \sum_{I=1}^{N_\sigma} \frac{1}{m_I} = \frac{1+2\sigma}{1+\sigma}.$$

- One additional latitudinal coordinate in even dimensions
- Part of the metric is similar to AdS_2

The conformal $SO(2,1)$ symmetry

$$\{H,D\}=H,\quad \{H,K\}=2D,\quad \{D,K\}=K,\qquad \mathcal{I}=HK-D^2$$

allows us to redefine r and its canonical conjugate momentum p_r so the Hamiltonian takes formally non-relativistic form[1]

$$H=\frac{1}{2}p_R^2+\frac{2\mathcal{I}(x,p_x,p_\varphi)}{R^2}.$$

- $R=\sqrt{2K}$, $p_R=\frac{2D}{\sqrt{2K}}$ are the “radius" and its canonical conjugate momentum
- $\mathcal{I}$ is the Casimir element of $SO(2,1)$

[1] **Hakobyan:2009ac**.

Some important consequences...

- The radial part of the Hamiltonian is separated.
- We just need to study Casimir of $SO(2,1)$, which is called angular mechanics.
- The variables φ_i are cyclic. Thus their canonically conjugate momenta p_{φ_i} are first integrals (in total N for both odd and even dimensions).

Fully non-isotropic NHEMP

Here we assume that none of the rotation parameters m_I are equal to each other

- Angular mechanics is

$$\mathcal{I} = A(x)\left[\sum_{a,b=1}^{N_\sigma-1} h^{ab}(x)p_a p_b + \sum_{i=1}^{N}\frac{p_{\varphi_i}^2}{x_i^2} + g_0(p_\varphi)\right]$$

- Separation of variables takes place in ellipsoidal coordinates

$$x_I^2 = (m_I - \lambda_I)\prod_{J\neq I}^{N_\sigma}\frac{m_I - \lambda_J}{m_I - m_J}$$

- Thus fully non-isotropic NHEMP is integrable for arbitrary higher dimensions with $N_\sigma + N + 1$ first integrals

Fully isotropic NHEM

Here we assume that all of the rotation parameters m_I are equal to each other. The angular mechanics is

$$\mathcal{I}_N = \sum_{i,j=1}^{N} (\eta^2(x)\delta_{ij} - x_i x_j) p_i p_j + \sum_{i=1}^{N} \frac{\eta^2(x) p_{\varphi_i}^2}{x_i^2} + \omega(p_{\varphi_i}) \sum_{i=1}^{N} x_i^2,$$

- In odd dimensions

  $$\eta^2 = N, \qquad \omega = 0$$

  The system is a generalization of Higgs oscillator, known as Rossochatius system.

- This is not the case in even dimensions.

Fully isotropic NHEM

- Both of the systems admit separation of variables by recursively introducing spherical coordinates

$$x_{N_\sigma} = \sqrt{N_\sigma}\cos\theta_{N_\sigma-1}, \quad x_a = \sqrt{N_\sigma}\tilde{x}_a \sin\theta_{N_\sigma-1}, \quad \sum_{a=1}^{N_\sigma-1} \tilde{x}_a^2 = 1,$$

- The systems also contain hidden symmetries, which
  - make the odd dimensional Rossochatius system maximally superintegrable
  - make the even dimensional system superintegrable (lacking one constant of motion to be maximally superintegrable)

Partially isotropic NHEMP

Let's discuss the simplest mixed case in odd $(2N+1)$ dimensions. We have p non-equal rotation parameters and l equal rotation parameters such that $p+l=N$

$$m_1 \neq m_2 \neq \ldots \neq m_p \neq \kappa, \quad m_{p+1} = m_{p+2} = \ldots = m_N \equiv \kappa.$$

- Non of the BH rotation parameters is 0.
- Separation of variables is achieved by introducing a mixture of spherical and ellipsoidal coordinates.

Partially isotropic NHEMP

If $l = 1$ the spherical subsystem is trivial and does not produce new integrals of motion. This is the fully non-isotropic integrable case.
If $l \geq 2$ then

- The $l - 1$ dimensional spherical subsystem is maximally superintegrable

$$\# \text{ of first integrals} \quad 2(l-1)-1$$

- The non-isotropic system contains p integrals of motion

$$\# \text{ of first integrals} \quad p + 2(l-1) - 1 = (N-1) + l - 2$$

In the fully isotropic ($p = 0,\ l = N$), the angular mechanics is maximally superintegrable with $2N - 3$ first integrals

NH limit of Extremal Vanishing Horizon MP BH (NHEVHMP)

NHEVHMP

NHEVHMP is obtained from the extremal MP metric by taking one of the rotation parameters equal to 0 and obtaining the NH limit. This results into a well defined solution of vacuum Einstein equations.

$$\frac{ds^2}{r_0^2} = F_0(x) ds^2_{AdS_3} + \sum_{a}^{N-1} dx_a^2 + \sum_{a,b}^{N-1} \tilde{\gamma}_{ab}(x)\, x_a x_b d\varphi_a d\varphi_b,$$

- Notice $ds^2_{AdS_3}$ term in the metric.
- The isometry contains $SO(2,1) \times SO(2,1)$ part.

NHEVHMP

- Although we have two conformal groups, they give rise to the same Casimir element. Thus we have a single angular mechanics and no additional constants of motion compared to non-EVH case.
- The rest of the discussion is the same for fully isotropic, fully non-isotropic and generic cases.

KG on NHEMP background

KG on NHEMP

Let's discuss Klein-Gordon field in the background of NHEMP black hole. We will bound the discussion to fully non-isotropic case in odd $(d = 2N + 1)$ dimensions.

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) = M^2\Phi\,,$$

As one would expect, the separation of variables takes place in elliptic coordinates as in the case of classical particles.

$$x_I^2 = (m_I - \lambda_I)\prod_{J\neq I}^{N_\sigma}\frac{m_I - \lambda_J}{m_I - m_J}$$

where NHEMP has the following form

$$\frac{ds^2}{r_H^2} = A(\lambda)\left(-r^2d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{a=1}^{N-1}h_a(\lambda)d\lambda_a^2 + \sum_{i,j=1}^{N}\tilde{\gamma}_{ij}x_i(\lambda)x_j(\lambda)D\varphi^iD\varphi^j$$

KG in NHEMP

After calculating the inverse metric and metric determinant, KG equation can be rewritten in the following form:

$$
\begin{aligned}
&\frac{1}{A(\lambda)}\left(-\frac{1}{r^2}\left[\frac{\partial}{\partial\tau}-\sum_{i=1}^{N} rk^i\frac{\partial}{\partial\varphi^i}\right]^2\Phi + r^2\partial_r^2\Phi + 2r\partial_r\Phi\right)\\
&+\sum_{a=1}^{N-1} h^a\partial_{\lambda_a}^2\Phi - \sum_{a=1}^{N-1}\sum_{i=1}^{N}\frac{h^a}{m_i-\lambda_a}\partial_{\lambda_a}\Phi\\
&+\sum_{i=1}^{N}\frac{1}{x_i^2}\partial_{\varphi_i}^2\Phi - \sum_{i,j=1}^{N}\frac{\sqrt{m_i-1}}{m_i}\frac{\sqrt{m_j-1}}{m_j}\partial_{\varphi_i}\partial_{\varphi_j}\Phi = M^2\Phi\,.
\end{aligned}
$$

The variables can be separated if we consider the following ansatz:

$$
\Phi = R_r(r)\cdot\prod_{a=1}^{N-1} R_{\lambda_a}(\lambda_a)\cdot e^{i\omega\tau}\cdot\prod_{b=1}^{N} e^{iL_b\varphi_b}\,,
$$

where ω and L_b are constants.

KG in NHEMP

The following equations describe the dynamics of r

$$r^2 \frac{R_r''}{R_r} + 2r \frac{R_r'}{R_r} + \frac{1}{r^2} \left(\omega - r \sum_{i=1}^{N} k^i L_i \right)^2 = \mathcal{C}_2 \, .$$

and λ_a

$$- \frac{4}{\lambda_a} \left(\frac{R_{\lambda_a}''}{R_{\lambda_a}} - \frac{R_{\lambda_a}'}{R_{\lambda_a}} \sum_{i}^{N} \frac{1}{m_i - \lambda_a} \right) \prod_{j=1}^{N} (m_j - \lambda_a)$$
$$+ \frac{b}{\lambda_a} \mathcal{C}_2 \prod_{i=1}^{N} m_i + (-1)^{N-1} \sum_{i=1}^{N} \frac{g_{\varphi_i}}{m_i - \lambda_a} + g_0 (-\lambda_a)^{N-2} = \sum_{\alpha=1}^{N-1} k_\alpha \lambda_a^{\alpha-1}.$$

where $\mathcal{C}_2$, k_α and g_{φ_i} are constants.

KG in NHEMP

After the following transformation,

$$z = \frac{2\,i\,\omega}{r}\,,$$

the radial equation becomes Whittaker's differential equation

$$\frac{d^2 R_r}{dz^2} + \left(-\frac{1}{4} + \frac{K}{z} + \frac{(1/4 - \mu^2)}{z^2} \right) R_r = 0\,,$$

where K and μ are constants related to k^i, L_i and $\mathcal{C}_2$.

- The general solutions to this equation are Whittaker's functions : $\mathcal{M}_{K,\mu}(z)$ and $\mathcal{W}_{K,\mu}(z)$ which can be expressed through confluent hypergeometric functions.
- Their behavior of Whittaker's functions at $r \to 0$ and $r \to \infty$ strongly depend on the values of k^i, L_i and $\mathcal{C}_2$ and can be used to put physical restrictions on these constants.

KG on NHEVHMP background

KG on NHEVHMP

Let's discuss Klein-Gordon field in the background of fully non-isotropic, odd dimensional ($d = 2N + 1$) NHEVHMP black hole.

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) = M^2\Phi\,,$$

As one would expect, the separation of variables takes place in elliptic coordinates as in the case of classical particles.

$$x_I^2 = (m_I - \lambda_I)\prod_{J\neq I}^{N_\sigma}\frac{m_I - \lambda_J}{m_I - m_J}$$

where NHEVHMP has the following form

$$\begin{aligned} ds^2 = \quad & F(\lambda)\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2 d\psi^2\right) \\ & + \sum_{a=1}^{N-1}\hat{h}_a d\lambda_a^2 + \sum_{a,b=1}^{N-1}\hat{\gamma}_{ab}\,\hat{x}_a(\lambda)\hat{x}_b(\lambda)d\varphi_a d\varphi_b \end{aligned}$$

KG on NHEVHMP

After calculating the inverse metric and metric determinant, KG equation can be rewritten in the following form:

$$
\begin{aligned}
&\frac{1}{F(\lambda)}\left(-\frac{\partial_\tau^2\Phi-\partial_\psi^2\Phi}{\rho^2}+\rho^2\partial_\rho^2\Phi+3\rho\partial_\rho\Phi\right)\\
&+\sum_{a=1}^{N-1}\frac{1}{\hat{x}_a^2}\partial_{\varphi_a}^2\Phi-\sum_{a,b}^{N-1}\frac{1}{\sqrt{m_a}}\frac{1}{\sqrt{m_b}}\partial_{\varphi_a}\partial_{\varphi_b}\Phi\\
&+\sum_{a=1}^{N-1}\hat{h}^a\partial_{\lambda_a}^2\Phi+\sum_{a=1}^{N-1}\frac{\hat{h}^a}{\lambda_a}\partial_{\lambda_a}\Phi-\sum_{a,b=1}^{N-1}\frac{\hat{h}^a}{m_b-\lambda_a}\partial_{\lambda_a}\Phi=M^2\Phi\,.
\end{aligned}
$$

The variables can be separated if we consider the following ansatz:

$$
\Phi=R_\rho(\rho)\cdot\prod_{a=1}^{N-1}R_a(\lambda_a)\cdot e^{i(-k_\tau\tau+m_\psi\psi)}\cdot\prod_{b=1}^{N-1}e^{iL_b\varphi_b}\,,
$$

where k_τ, m_ψ and L_b are constants.

KG on NHEVHMP

The dynamics of ρ is described by

$$\left(\frac{k_\tau^2 - m_\psi^2}{\rho^2} + \rho^2\partial_\rho^2 + 3\,\rho\,\partial_\rho\right) R_\rho(\rho) = -4\,\hat{\mathcal{C}}_2\,R_\rho(\rho).$$

And for λ_a we have

$$4\left(\frac{R_a''}{R_a} + \frac{1}{\lambda_a}\frac{R_a'}{R_a} - \sum_{b=1}^{N-1}\frac{R_a'/R_a}{m_b-\lambda_a}\right)\prod_{c=1}^{N-1}(m_c-\lambda_a)$$
$$-\frac{4\hat{\mathcal{C}}_2}{\lambda_a}\prod_{b=1}^{N-1}m_b + \sum_{b=1}^{N-1}\frac{\hat{q}_{\varphi_b}}{m_b-\lambda_a} + \hat{q}_0(-\lambda_a)^{N-2} = \sum_{\alpha=1}^{N-1}k_\alpha\lambda_a^{\alpha-1}$$

where $\hat{q}_{\varphi_b}$, $\hat{q}_0$ and k_α are constants.

KG on NHEVHMP

After the following transformation

$$R_\rho(\rho) = \frac{u(\rho)}{\rho}, \quad \text{and} \quad \rho = \frac{\sqrt{k_\tau^2 - m_\psi^2}}{z},$$

the radial differential equation becomes Bessel's equation

$$z^2 \frac{d^2}{dz^2} u + z \frac{d}{dz} u + \left(z^2 - \nu^2\right) u = 0, \quad \nu^2 = 1 - 4\hat{\mathcal{C}}_2$$

- The general solutions to this equation are Bessel function $J_\nu(z)$, $Y_\nu(z)$.
- Their behavior of these functions at $r \to 0$ and $r \to \infty$ restrict the value of $\mathcal{C}_2$.

Thank you!

[1] H. Demirchian, S. Sadeghian "Separability of Klein-Gordon Equation on Near Horizon Extremal Myers-Perry Black Hole," to be published

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[3] H. Demirchian, A. Nersessian, S. Sadeghian, and M. Sheikh-Jabbari, "Integrability of geodesics in near-horizon extremal geometries: Case of Myers-Perry black holes in arbitrary dimensions," *Phys. Rev. D* **97** (may, 2018) 104004.

[4] A. Galajinsky, A. Nersessian, and A. Saghatelian, "Superintegrable models related to near horizon extremal Myers-Perry black hole in arbitrary dimension," 1303.4901.

[5] H. Demirchian, "Note on constants of motion in conformal mechanics associated with near horizon extremal Myers–Perry black holes," *Mod. Phys. Lett. A* **32** (2017), no. 27, 1750144, 1706.04861.

[6] T. Hakobyan, S. Krivonos, O. Lechtenfeld, and A. Nersessian, "Hidden symmetries of integrable conformal mechanical systems," 0908.3290.

[7] T. Hakobyan, O. Lechtenfeld, A. Nersessian, and A. Saghatelian, "Invariants of the spherical sector in conformal mechanics," 1008.2912.

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Equations

NHEMP geometry

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(x,\sigma) x_i x_j D\varphi^i D\varphi^j$$

where

$$N_\sigma = N + \sigma, \qquad \sigma = \begin{cases} 0 & \text{when} \quad D = 2N+1 \\ 1 & \text{when} \quad D = 2N+2 \end{cases},$$

$$A(x) = \frac{\sum_{I=1}^{N_\sigma} x_I^2/m_I^2}{\frac{\sigma}{1+\sigma} + 4\sum_{i<j}^{N} \frac{1}{m_i}\frac{1}{m_j}},$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_I^{N_\sigma} x_I^2/m_I^2} \frac{\sqrt{m_i - 1}x_i}{m_i} \frac{\sqrt{m_j - 1}x_j}{m_j}$$

First integrals of fully-non isotropic NHEMP

$$F_a(x,\sigma) = K^{bc}_{(a)}(x,\sigma)\ p_b p_c + L^{ij}_{(a)}(x,\sigma)\ p_{\varphi_i} p_{\varphi_j} + A_{(a)}(x,\sigma) m_0^2 r_H^2$$

where

$$K^{bc}_{(a)} = \left(\sum_{\alpha=0}^{N_\sigma - a - 1} (-1)^{N_\sigma + \alpha - a} A_\alpha m_b^{N_\sigma - \alpha - a} + x_b^2 \sum_{\alpha=1}^{N_\sigma - a - 1} (-1)^\alpha M^{\neq b}_{N_\sigma - \alpha - a - 1} m_b^\alpha \right) \delta^{bc} + M^{\neq b,c}_{N_\sigma - a - 1} x_b x_c$$

$$L^{ij}_{(a)} = \left((1 - \delta^1_a) \sum_{\alpha=1}^{N_\sigma - a} (-1)^{N_\sigma + \alpha} A_{\alpha - 1} m_i^{N_\sigma - a - \alpha + 1} - \delta^1_a A_{N_\sigma - 1} \right) \frac{\delta^{ij}}{x_i^2} + (-1)^{a-1} A_{N_\sigma - a} \frac{\sqrt{m_i - 1}}{m_i} \frac{\sqrt{m_j - 1}}{m_j}$$