Elliptic, hyperbolic, complex gamma functions and physics in the various dimensions.

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and
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Elliptic gamma function

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^j q^k}\,, \quad |p|, |q| < 1\,, \quad z \in \mathbb{C}^*. \tag{0.1}$$

$$\Gamma(z;p,q) = \Gamma(z;q,p) \tag{0.2}$$

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;q,p) \quad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;q,p) \tag{0.3}$$

Let us take definition of the Dedekind η-function and Jacobi θ_1-function

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}(e^{2\pi i\tau}; e^{2\pi i\tau})_\infty \tag{0.4}$$

where $(a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$, and for $q = e^{2\pi i\tau}$

$$\theta_1(u|\tau) = -\theta_{11}(u) = -\sum_{\ell\in\mathbb{Z}+1/2} e^{\pi i\tau\ell^2} e^{2\pi i\ell(u+1/2)} =$$

$$iq^{1/8} e^{-\pi iu}(q; q)_\infty \theta(e^{2\pi iu}; q), \tag{0.5}$$

with

$$\theta(z; q) = (z; q)_\infty (qz^{-1}; q)_\infty. \tag{0.6}$$

Elliptic beta integral

$$\prod_{a=1}^{6} t_a = pq \tag{0.7}$$

$$\kappa \int_{\mathbb{T}} \frac{\prod_{a=1}^{6} \Gamma(t_a z; p, q)\Gamma(t_a z^{-1}; p, q)}{\Gamma(z^2; p, q)\Gamma(z^{-2}; p, q)} \frac{dz}{z} = \prod_{1 \le a < b \le 6} \Gamma(t_a t_b; p, q). \tag{0.8}$$

where $\mathbb{T}$ is the unit circle of positive orientation and

$$\kappa = \frac{(p; p)_\infty (q; q)_\infty}{4\pi i}. \tag{0.9}$$

V-function

$$\prod_{a=1}^{8} t_a = p^2 q^2 \tag{0.10}$$

$$V(t_1,\ldots,t_8;p,q) = \kappa \int_{\mathbb{T}} \frac{\prod_{a=1}^{8} \Gamma(t_a z;p,q)\Gamma(t_a z^{-1};p,q)}{\Gamma(z^2;p,q)\Gamma(z^{-2};p,q)} \frac{dz}{z}. \tag{0.11}$$

$W(E_7)$-group transformation laws:I

$$V(t_1,\ldots,t_8;p,q) = \prod_{1\leq j<k\leq 4} \Gamma(t_jt_k;p,q) \times \tag{0.12}$$

$$\prod_{5\leq j<k\leq 8} \Gamma(t_jt_k;p,q) V(s_1,\ldots,s_8;p,q)$$

$$s_j = \rho^{-1} t_j, \quad j = 1,2,3,4 \tag{0.13}$$

$$s_j = \rho t_j, \quad j = 5,6,7,8 \tag{0.14}$$

$$\rho = \sqrt{\frac{t_1t_2t_3t_4}{pq}} = \sqrt{\frac{pq}{t_5t_6t_7t_8}} \tag{0.15}$$

$W(E_7)$-group transformation laws:II

$$V(t_1, \ldots, t_8; p, q) = \prod_{j,k=1}^{4} \Gamma(t_j t_{k+4}; p, q) \times \tag{0.16}$$
$$V(T^{1/2}/t_1, \ldots, T^{1/2}/t_4, U^{1/2}/t_5, \ldots, U^{1/2}/t_8; p, q)$$

where $T = t_1 t_2 t_3 t_4$ and $U = t_5 t_6 t_7 t_8$.

$W(E_7)$-group transformation laws:III

$$V(t_1,\ldots,t_8;p,q) = \prod_{1\leq j<k\leq 8} \Gamma(t_j t_k;p,q) \times \quad (0.17)$$
$$V(\sqrt{pq}/t_1,\ldots,\sqrt{pq}/t_8;p,q)$$

4D Seiberg duality

The left-hand side of the univariate elliptic beta integral evaluation formula (0.8) describes the superconformal index of the supersymmetric quantum chromodynamics with $SU(2)$ gauge group and $SU(6)$ flavor group. This theory has one vector superfield (gauge fields) in the adjoint representation of $SU(2)$ and a set of chiral superfields (matter fields) in the fundamental representation of $SU(2) \times SU(6)$. In the deep infrared region the theory is strongly coupled, all colored particles confine, and one has the Wess-Zumino type model for mesonic fields lying in the 15-dimensional totally antisymmetric tensor representation of $SU(6)$. The superconformal index of the latter theory is described by the right-hand side expression of formula (0.8).

$$\lim_{v\to 0}\Gamma(e^{-2\pi v y}; e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) = e^{\frac{-\pi(2y-\omega_1-\omega_2)}{12v\omega_1\omega_2}}\gamma^{(2)}(y;\omega_1,\omega_2),$$

where $\gamma^{(2)}(y;\omega_1,\omega_2)$ is the hyperbolic gamma function.

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the integral representation

$$\gamma^{(2)}(y;\omega_1,\omega_2) = \exp\left(-\int_0^\infty\left(\frac{\sinh(2y-\omega_1-\omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)} - \frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2 x}\right)\right)$$

and obeys the equations:

$$\frac{\gamma^{(2)}(y+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_2}\,, \quad \frac{\gamma^{(2)}(y+\omega_2;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_1}.$$

$$\gamma^{(2)}(u;\omega_1,\omega_2) = e^{-\frac{i\pi}{2}B_{2,2}(u,\omega_1,\omega_2)}\gamma(u;\omega_1,\omega_2)$$

where $B_{2,2}(y;\omega_1,\omega_2)$ is the second order Bernoulli polynomial:

$$B_{2,2}(y;\omega_1,\omega_2) = \frac{y^2}{\omega_1\omega_2} - \frac{y}{\omega_1} - \frac{y}{\omega_2} + \frac{1}{6}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) + \frac{1}{2}.$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the following asymptotics :

$$\lim_{y\to\infty} e^{\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2)=1,\ \arg\omega_1<\arg y<\arg\omega_2+\pi,$$

$$\lim_{y\to\infty} e^{-\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2)=1,\ \arg\omega_1-\pi<\arg y<\arg\omega_2,$$

where $B_{2,2}(y;\omega_1,\omega_2)$ is the second order Bernoulli polynomial.

$$\sum \mu_k = \omega_1 + \omega_2 = Q \tag{0.18}$$

$$\frac{1}{2}\int_{-i\infty}^{i\infty} \frac{\prod_1^6 \gamma^{(2)}(\mu_k+z)\gamma^{(2)}(\mu_k-z)}{\gamma^{(2)}(2z)\gamma^{(2)}(-2z)}\frac{dz}{i\sqrt{\omega_1\omega_2}} = \prod_{1\leq a<b\leq 6}\gamma^{(2)}(\mu_a+\mu_b) \tag{0.19}$$

$W(E_7)$-group transformation laws: H-I

$$\sum_{i=1}^{8} \mu_i = 2Q \tag{0.20}$$

$$I_h(\mu_i) = \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{8} \gamma^{(2)}(\mu_i \pm z)}{\gamma^{(2)}(\pm 2z)} \frac{dz}{i\sqrt{\omega_1\omega_2}} \tag{0.21}$$

$$I_h(\mu_i) = \prod_{1\leq i<j\leq 4} \gamma^{(2)}(\mu_i + \mu_j) \prod_{5\leq i<j\leq 8} \gamma^{(2)}(\mu_i + \mu_j) I_h(\nu_i) \tag{0.22}$$

$$\xi = \frac{1}{2}(Q - \sum_{i=1}^{4} \mu_i) \tag{0.23}$$

$$\nu_i = \mu_i + \xi \quad i = 1,2,3,4 \quad \nu_i = \mu_i - \xi \quad i = 5,6,7,8 \tag{0.24}$$

$W(E_7)$-group transformation laws: H-II

$$M_1 = \frac{1}{2}\sum_{i=1}^{4}\mu_i \quad M_2 = \frac{1}{2}\sum_{i=5}^{8}\mu_i \tag{0.25}$$

$$\nu_i = M_1 - \mu_i \quad i = 1,2,3,4 \quad \nu_i = M_2 - \mu_i \quad i = 5,6,7,8 \tag{0.26}$$

$$I_h(\mu_i) = \prod_{j,k=1}^{4}\gamma^{(2)}(\mu_j + \mu_{k+4}) I_h(\nu_i) \tag{0.27}$$

$W(E_7)$-group transformation laws: H-III

$$I_h(\mu_i) = \prod_{1\leq i<j\leq 8} \gamma^{(2)}(\mu_i+\mu_j) I_h\left(\frac{Q}{2}-\mu_i\right) \tag{0.28}$$

$$\mu_k = f_k + i\nu \quad k = 1,2,3 \quad \mu_k = g_k - i\nu \quad k = 4,5,6 \quad z \rightarrow z - i\nu \tag{0.29}$$

$$\int \frac{dx}{i\sqrt{\omega_1\omega_2}} \prod_{i=1}^{3} \gamma^{(2)}(x+f_i)\gamma^{(2)}(-x+g_i) = \prod_{i,j=1} \gamma^{(2)}(f_i+g_j) \tag{0.30}$$

$$\sum_i (f_i + g_i) = Q \tag{0.31}$$

$$\gamma^{(2)}(x, b, b^{-1}) = S_b(x) \tag{0.32}$$

$$\int \frac{dx}{i} \prod_{i=1}^{3} S_b(x+f_i) S_b(-x+g_i) = \prod_{i,j=1} S_b(f_i+g_j) \tag{0.33}$$

$$\sum_i (f_i+g_i) = Q \tag{0.34}$$

$$f_3 \to i\infty \quad g_3 = Q - f_1 - f_2 - f_3 - g_1 - g_2 \tag{0.35}$$

$$\int_{-i\infty}^{i\infty} \exp\left(i\pi\left[(y(f_1+f_2+g_1+g_2)+f_1 f_2 - g_1 g_2\right]\right)$$

$$S_b(y+f_1)S_b(y+f_2)S_b(-y+g_1)S_b(-y+g_2)\frac{dy}{i} =$$

$$S_b(Q-f_1-f_2-g_1-g_2)S_b(f_1+g_1)S_b(f_1+g_2) \times$$

$$S_b(f_2+g_1)S_b(f_2+g_2).$$

Pentagon identity

$$f_2 \to -i\infty \quad g_2 \to i\infty \quad f_2 + g_2 = \alpha \tag{0.36}$$

$$\int_{-i\infty}^{i\infty} \exp i\pi(y(2\alpha + g - Q)) S_b(y) S_b(-y + g) \frac{dy}{i}$$
$$= \exp i\pi(g(\alpha - Q/2) + g^2/2) S_b(Q - \alpha - g) S_b(\alpha) S_b(g).$$

Example of 3D Seiberg duality: Mirror symmetry between $N = 2$ SQED with two chiral multiplets and XYZ model on squashed sphere:

$$b^2(x_0^2 + x_1^2) + b^{-2}(x_3^2 + x_4^2) = 1$$

6j-symbols and fusion matrix

$$J_h(g_a, f_a) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} \gamma^{(2)}(z+f_a, ; \omega_1, \omega_2)\gamma^{(2)}(-z+g_a, ; \omega_1, \omega_2) \frac{dz}{i\sqrt{\omega_1\omega_2}}$$

$6j$-symbols of the Faddeev modular quantum double:
$U_q(sl(2,\mathbb{R}) \oplus U_{\tilde{q}}(sl(2,\mathbb{R}),\ q = e^{\pi i b^2}$ and $\tilde{q} = e^{\pi i b^{-2}}$.

$$\left\{ \begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_b = \frac{S_b(\alpha_s + \alpha_2 - \alpha_1) S_b(\alpha_1 + \alpha_t - \alpha_4)}{S_b(\alpha_t + \alpha_2 - \alpha_3) S_b(\alpha_3 + \alpha_s - \alpha_4)} |S_b(2\alpha_t)|^2 J_h(\mu, \nu)$$

$$\nu_1 = \alpha_s + \alpha_1 - \alpha_2 \tag{0.37}$$

$$\nu_2 = Q + \alpha_s - \alpha_1 - \alpha_2 \tag{0.38}$$

$$\nu_3 = \alpha_s + \alpha_3 - \alpha_4 \tag{0.39}$$

$$\nu_4 = Q + \alpha_s - \alpha_3 - \alpha_4 \tag{0.40}$$

$$\mu_1 = -Q - \alpha_s + \alpha_t + \alpha_4 + \alpha_2 \tag{0.41}$$

$$\mu_2 = -\alpha_s - \alpha_t + \alpha_4 + \alpha_2 \tag{0.42}$$

$$\mu_3 = Q - 2\alpha_s \tag{0.43}$$

$$\mu_4 = 0 \tag{0.44}$$

6j-symbols symmetries

$$J_h(\underline{\mu},\underline{\nu}) = \prod_{j,k=1}^{2} \gamma^{(2)}(\mu_j+\nu_k;\omega_1,\omega_2) \prod_{j,k=3}^{4} \gamma^{(2)}(\mu_j+\nu_k;\omega_1,\omega_2) \quad (0.4$$

$$\times J_h(\mu_1+\eta,\mu_2+\eta,\mu_3-\eta,\mu_4-\eta,$$

$$\nu_1+\eta,\nu_2+\eta,\nu_3-\eta,\nu_4-\eta),$$

where

$$\eta = \frac{1}{2}(\omega_1+\omega_2-\mu_1-\mu_2-\nu_1-\nu_2). \qquad (0.46)$$

$$J(\mu,\nu) = \tag{0.47}$$

$$\prod_{j,k=1}^{2} \gamma^{(2)}(\nu_j + \mu_{k+2}; \omega_1, \omega_2) \prod_{j,k=1}^{2} \gamma^{(2)}(\nu_{j+2} + \mu_k; \omega_1, \omega_2) \times$$

$$J(G-\nu_1, G-\nu_2, Q-G-\nu_3, Q-G-\nu_4;$$

$$G-\mu_1, G-\mu_2, Q-G-\mu_3, Q-G-\mu_4)$$

where

$$J(\mu,\nu) = \int_{-i\infty}^{i\infty} \prod_{i=1}^{4} \gamma^{(2)}(z+\nu_a; \omega_1, \omega_2) \prod_{i=1}^{4} \gamma^{(2)}(-z+\mu_a; \omega_1, \omega_2) dz \tag{0.48}$$

and $G = \frac{1}{2}(\nu_1 + \nu_2 + \mu_1 + \mu_2)$.

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{4} \gamma^{(2)}(f_j + z;\omega)\gamma^{(2)}(g_j - z;\omega) dz = \prod_{j,k=1}^{4} \gamma^{(2)}(g_j + f_k;\omega)$$
$$\times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{4} \gamma^{(2)}(\tfrac{1}{2}Q - f_j + z;\omega)\gamma^{(2)}(\tfrac{1}{2}Q - g_j - z;\omega) dz.$$

Complex hypergeometric functions

$$\mathbf{\Gamma}(x,n) == \frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1+\frac{n-\mathrm{i}x}{2})}, \quad n \in \mathbb{Z} \tag{0.50}$$

$$\mathbf{\Gamma}(x,-n) = (-1)^n \mathbf{\Gamma}(x,n) \tag{0.51}$$

$$\mathbf{\Gamma}(x,n)\mathbf{\Gamma}(-x-2\mathrm{i},n) = 1 \tag{0.52}$$

$$\mathbf{\Gamma}(x-2\mathrm{i},n) = \frac{n^2+x^2}{4}\mathbf{\Gamma}(x,n) \tag{0.53}$$

Complex hypergeometric functions as limits from hyperbolic integrals

Now set

$$\sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta, \quad \delta \to 0^+.$$

Then $\omega_1 + \omega_2 = 2\delta\sqrt{\omega_1\omega_2} + O(\delta^2) \to 0$ and

$$\sqrt{\frac{\omega_2}{\omega_1}} = -\mathrm{i}+\delta+O(\delta^2), \quad \frac{\omega_1}{\omega_2} = -1+2\mathrm{i}\delta+\delta^2, \quad \frac{\omega_2}{\omega_1} = -1-2\mathrm{i}\delta+O(\delta^2).$$

Special choice of the argument u:

$$u = \mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta), \quad n \in \mathbb{Z},\ x \in \mathbb{C}.$$

One can show that in this limit uniformly on the compacta

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta); \omega_1, \omega_2) \to (4\pi\delta)^{\mathrm{i}x-1} e^{\frac{\pi \mathrm{i}}{2}n^2} \mathbf{\Gamma}(x, n).$$

Degeneration of the integrals.

Consider

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Delta(z) \frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}} = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Delta(\sqrt{\omega_1\omega_2}x)\frac{dx}{\mathrm{i}}, \quad x = \frac{z}{\sqrt{\omega_1\omega_2}},$$

where $\Delta(z) \propto$ a product of $\gamma^{(2)}(u;\omega_1,\omega_2)$.

$$\gamma^{(2)}(\lambda u; \lambda\omega_1, \lambda\omega_2) = \gamma^{(2)}(u;\omega_1,\omega_2), \quad \lambda \neq 0.$$

Therefore $\omega_1\omega_2$ is any nonzero number: in QFT $\omega_1\omega_2 = 1$.
Rewrite

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Delta(\sqrt{\omega_1\omega_2}\,x)\frac{dx}{\mathrm{i}} = \sum_{N\in\mathbb{Z}} \int_{\mathrm{i}(N-1/2)}^{\mathrm{i}(N+1/2)} \Delta(\sqrt{\omega_1\omega_2}\,x)\frac{dx}{\mathrm{i}}$$

$$= \sum_{N\in\mathbb{Z}} \int_{N-1/2}^{N+1/2} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}\,x)dx = \sum_{N\in\mathbb{Z}} \int_{-1/2}^{1/2} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+x))dx.$$

Parametrise $x = y\delta$, $\delta > 0$, and take the limit $\delta \to 0^+$

$$\sum_{N\in\mathbb{Z}} \int_{-1/2}^{1/2} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+x))dx$$

$$= \lim_{\delta\to 0} \sum_{N\in\mathbb{Z}} \int_{-1/2\delta}^{1/2\delta} \delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))dy.$$

The sum over N is infinite, for $\delta \to 0^+$ the integration contour becomes $(-\infty,\infty)$. Uniformness of the limit $\Rightarrow$

$$= \sum_{N\in\mathbb{Z}} \int_{-\infty}^{\infty} \left[\lim_{\delta\to 0}\delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))\right] dy.$$

Apply this limit to the general univariate hyperbolic beta integral.

Parametrization of the integral kernel for (0.19):

$$z = \mathrm{i}\sqrt{\omega_1\omega_2}(N+\delta y), \quad y \in \mathbb{C}, \quad N \in \mathbb{Z}+\nu, \ \nu = 0, \frac{1}{2},$$

$$\mu_k = \mathrm{i}\sqrt{\omega_1\omega_2}(N_k+\delta \alpha_k), \quad \alpha_k \in \mathbb{C}, \quad N_k \in \mathbb{Z}+\nu, \ \nu = 0, \frac{1}{2},$$

The limit $\delta \to 0^+ \ \Rightarrow$ the balancing condition

$$\sum_{k=1}^{6} \alpha_k = -2\mathrm{i}, \quad \sum_{k=1}^{6} N_k = 0.$$

The parameter $\nu = 0, 1/2$ emerges because only the sums $N_k \pm N$ should be integers. Now, $\delta \to 0^+$ limiting relations:

$$\prod_{k=1}^{6} \gamma^{(2)}(\mu_k \pm z; \omega) \to \frac{(-1)^{2\nu}}{(4\pi\delta)^8} \prod_{k=1}^{6} \mathbf{\Gamma}(\alpha_k + y, N_k + N)\mathbf{\Gamma}(\alpha_k - y, N_k - N),$$

$$\prod_{1\le j<k\le 6}\gamma^{(2)}(\mu_j+\mu_k;\omega)\to\frac{(-1)^{2\nu}}{(4\pi\delta)^5}\prod_{1\le j<k\le 6}\boldsymbol{\Gamma}(\alpha_j+\alpha_k,N_j+N_k),$$

$$\gamma^{(2)}(\pm 2z;\omega)\to\frac{(-1)^{2\nu}}{(4\pi\delta)^2}\frac{\Gamma(N+\mathrm{i}y)}{\Gamma(1+N-\mathrm{i}y)}\frac{\Gamma(-N-\mathrm{i}y)}{\Gamma(1-N+\mathrm{i}y)}=\frac{(4\pi\delta)^{-2}}{y^2+N^2}.$$

The diverging factors $(4\pi\delta)^{-5}$ on both sides cancel.
The final complex beta integral evaluation formula:

$$\frac{1}{8\pi}\sum_{N\in\mathbb{Z}+\nu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{6}\boldsymbol{\Gamma}(\alpha_k\pm y,N_k\pm N)dy$$

$$=\prod_{1\le j<k\le 6}\boldsymbol{\Gamma}(\alpha_j+\alpha_k,N_j+N_k),$$

where $\sum_{k=1}^{6}\alpha_k=-2\mathrm{i}$, $\sum_{k=1}^{6}N_k=0$, and

$$\boldsymbol{\Gamma}(x_1 \pm x_2, n_1 \pm n_2) := \boldsymbol{\Gamma}(x_1 + x_2, n_1 + n_2)\boldsymbol{\Gamma}(x_1 - x_2, n_1 - n_2)\,.$$

Degeneration of the $W(E_7)$-group transformation laws:

$$\sum_{N\in\mathbb{Z}+\nu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{8}\boldsymbol{\Gamma}(\alpha_k \pm y, N_k \pm N)dy$$

$$= (-1)^L \prod_{1\le j<k\le 4}\boldsymbol{\Gamma}(\alpha_j+\alpha_k, N_j+N_k)\prod_{5\le j<k\le 8}\boldsymbol{\Gamma}(\alpha_j+\alpha_k, N_j+N_k)$$

$$\times \sum_{N\in\mathbb{Z}+\mu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{4}\boldsymbol{\Gamma}(\alpha_k \pm y - \tfrac{1}{2}X - \mathrm{i}, N_k \pm N - \tfrac{1}{2}L)$$

$$\times \prod_{k=5}^{8}\boldsymbol{\Gamma}(\alpha_k \pm y + \tfrac{1}{2}X + \mathrm{i}, N_k \pm N + \tfrac{1}{2}L)dy,$$

with $X := \sum_{j=1}^{4} \alpha_j, L := \sum_{j=1}^{4} N_j$ and the balancing

$$\sum_{k=1}^{8} \alpha_k = -4\mathrm{i}, \quad \alpha_k \in \mathbb{C}, \quad \sum_{k=1}^{8} N_k = 0, \quad N_k \in \mathbb{Z} + \nu,\ \nu = 0, \frac{1}{2}.$$

6j-symbols of $SL(2,\mathbb{C})$

$$\left\{\begin{matrix} \sigma_1,N_1 & \sigma_2,N_2 \\ \sigma_3,N_3 & \sigma_4,N_4 \end{matrix}\middle|\begin{matrix} \rho_1,M_1 \\ \rho_2,M_2 \end{matrix}\right\} = \frac{\pi^2}{4}\frac{\boldsymbol{\Gamma}\left(\sigma_1-\sigma_2+\rho_2-i,\,\frac{N_1-N_2+M_2}{2}\right)}{\boldsymbol{\Gamma}\left(-\sigma_3-\sigma_4+\rho_2-i,\,\frac{-N_3-N_4+M_2}{2}\right)}$$

$$\times\frac{\boldsymbol{\Gamma}\left(\sigma_2-\sigma_3+\rho_1-i,\,\frac{N_2-N_3+M_1}{2}\right)}{\boldsymbol{\Gamma}\left(\sigma_1+\sigma_4+\rho_1-i,\,\frac{N_1+N_4+M_1}{2}\right)}$$

$$\times(-)^{(-N_2+M_2+N_4)}\mathcal{J}(R_i,S_i;U_i,T_i)$$

$$\mathcal{J}(R_i, S_i; U_i, T_i) = \sum_N \int dy \prod_{i=1}^{4} \mathbf{\Gamma}(R_i - y, -N + S_i)\mathbf{\Gamma}(U_i + y, N + T_i) \tag{0.54}$$

$$R_1 = -\sigma_1 + \sigma_2 - \rho_2 - i, \qquad S_1 = \frac{-N_1 + N_2 - M_2}{2} \tag{0.55}$$

$$R_2 = \sigma_1 + \sigma_2 - \rho_2 - i, \qquad S_2 = \frac{N_1 + N_2 - M_2}{2} \tag{0.56}$$

$$R_3 = -\sigma_3 - \sigma_4 - \rho_2 - i, \qquad S_3 = -\frac{N_3 + N_4 + M_2}{2} \tag{0.57}$$

$$R_4 = \sigma_3 - \sigma_4 - \rho_2 - i, \qquad S_4 = \frac{N_3 - N_4 - M_2}{2} \tag{0.58}$$

$$U_1 = -\rho_1 - \sigma_2 + \sigma_4 + \rho_2, \qquad T_1 = \frac{-M_1 - N_2 + N_4 + M_2}{2} \tag{0.59}$$

$$U_2 = \rho_1 - \sigma_2 + \sigma_4 + \rho_2, \qquad T_2 = \frac{M_1 - N_2 + N_4 + M_2}{2} \tag{0.60}$$

$$U_3 = 0, \qquad T_3 = 0 \tag{0.61}$$

$$U_4 = 2\rho_2, \qquad T_4 = M_2 \tag{0.62}$$

$$
\begin{aligned}
&\mathcal{J}(\underline{s},\underline{n};\underline{t},\underline{m}) = e^{\pi \mathrm{i} A}\prod_{j,k=1}^{2}\boldsymbol{\Gamma}(s_j+t_k,n_j+m_k)\prod_{j,k=3}^{4}\boldsymbol{\Gamma}(s_j+t_k,n_j+m_k)\\
&\times\mathcal{J}(s_1+Y,n_1+K,s_2+Y,n_2+K,s_3-Y,n_3-K,s_4-Y,n_4-K;\\
&t_1+Y,m_1+K,t_2+Y,m_2+K,t_3-Y,m_3-K,t_4-Y,m_4-K),\\
&K=-\frac{n_1+n_2+m_1+m_2}{2},\qquad Y=-\frac{s_1+s_2+t_1+t_2+2\mathrm{i}}{2},\\
&A=(n_1+n_2)(m_1+m_2)+(n_3+n_4)(m_3+m_4)+\\
&+\frac{1}{2}(1-(-1)^{2K})\left(1+\sum_{a=1}^{4}m_a\right).
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{N\in\mathbb{Z}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{k=1}^{4}\boldsymbol{\Gamma}(s_k+y,N_k+N)\boldsymbol{\Gamma}(t_k-y,M_k-N)dy\\
&=(-1)^{\sum_{k=1}^{4}N_k}\prod_{j,k=1}^{4}\boldsymbol{\Gamma}(s_j+t_k,N_j+M_k) \qquad (0.64)\\
&\times\sum_{N\in\mathbb{Z}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{k=1}^{4}\boldsymbol{\Gamma}(-\mathrm{i}-t_k+y,N-M_k)\boldsymbol{\Gamma}(-\mathrm{i}-s_k-y,-N-N_k)dy
\end{aligned}
$$