Non-compact complex projective spaces as a phase space of integrable systems: supersymmetric extensions

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Content

Based on

E.Kh., S.Krivonos, A.Nersessian, *"Kähler geometry for $su(1, N|M)$-superconformal mechanics"*, [arXiv:2110.11711].

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- Basic facts on Kähler supermanifolds.
- non-compact complex projective superspaces $\widetilde{\mathbb{CP}}^{N|M}$ in the parametrization similar to those of Klein model.
- Symmetry algebra of $\widetilde{\mathbb{CP}}^{N|M}$, $su(1,N|M)$-superconformal mechanics.
- Canonical coordinates.
- Superintegrable supergeneralizations of oscillator- and Coulomb-like systems.
- Conclusion.

Kähler Supermanifolds

An (even) $(N|M)$-dimensional Kähler supermanifold can be defined as a complex supermanifold with symplectic structure given by the expression

$$\Omega = \imath(-1)^{p_I(p_J+1)} g_{I\bar{J}} dZ^I \wedge d\bar{Z}^J, \quad d\Omega = 0, \tag{1}$$

The "matrix components" $g_{I\bar{J}} = \frac{\partial^L}{\partial Z^I} \frac{\partial^R}{\partial \bar{Z}^J} K(Z, \bar{Z})$,

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The "matrix components" $g_{I\bar{J}} = \frac{\partial^L}{\partial Z^I}\frac{\partial^R}{\partial \bar{Z}^J} K(Z, \bar{Z})$, The Poisson brackets associated with this Kähler structure looks as follows

$$\{f, g\} = \imath \left(\frac{\partial^R f}{\partial \bar{Z}^I} g^{\bar{I}J} \frac{\partial^L g}{\partial Z^J} - (-1)^{p_I p_J} \frac{\partial^R f}{\partial Z^I} g^{\bar{J}I} \frac{\partial^L g}{\partial \bar{Z}^J} \right) \tag{2}$$

where $\quad g^{\bar{I}J} g_{J\bar{K}} = \delta^{\bar{I}}_{\bar{K}}, \quad \overline{g^{\bar{I}J}} = (-1)^{p_I p_J} g^{\bar{J}I}.$

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

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Let us equip, at first, the $(N+1|M)$-dimensional complex superspace with the canonical symplectic structure

$$\Omega_0 = \imath \sum_{a,b=0}^{N} \gamma_{a\bar{b}} dv^a \wedge d\bar{v}^b + \sum_{A=1}^{M} d\eta^A \wedge d\bar{\eta}^A, \tag{3}$$

with $v^a, \bar{v}^a$ being bosonic, and $\eta^A, \bar{\eta}^A$ being fermionic variables.

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The matrix $\gamma_{a\bar{b}}$ is chosen in the form

$$\gamma_{a\bar{b}} = \left(\begin{array}{cc|ccc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right), \qquad a, b = N, 0, 1, ..., N-1. \qquad (4)$$

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With this supersymplectic structure we can associate the Poisson brackets given by the relations

$$\{v^a, \bar{v}^b\} = -\imath\gamma^{\bar{b}a}, \quad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}}, \qquad \gamma^{\bar{a}b}\gamma_{b\bar{c}} = \delta^{\bar{a}}_{\bar{c}}. \quad (5)$$

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$$\text{Equivalently,} \quad \{v^0, \bar{v}^N\} = 1, \quad \{v^N, \bar{v}^0\} = -1, \tag{6}$$

$$\{v^\alpha, \bar{v}^\beta\} = \imath\delta^{\alpha\bar{\beta}}, \quad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}} \tag{7}$$

Here we introduced the indices $\alpha, \beta = 1, \ldots, N-1$.

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

On this superspace we can define the linear Hamiltonian action of $u(N.1|M) = u(1) \times su(N.1|M)$ superalgebra

$$\{h_{a\bar{b}}, h_{c\bar{d}}\} = -\imath\left(h_{a\bar{d}}\gamma^{\bar{c}b} - h_{c\bar{b}}\gamma^{\bar{a}d}\right), \quad \{\Theta_{A\bar{a}}, h_{b\bar{c}}\} = -\imath\Theta_{A\bar{c}}\gamma^{\bar{b}a}, \tag{8}$$

$$\{\Theta_{A\bar{a}}, \bar{\Theta}_{\bar{B}b}\} = h_{b\bar{a}}\delta^{B\bar{A}} - R_{A\bar{B}}\gamma^{\bar{b}a}, \quad \{\Theta_{A\bar{a}}, R_{C\bar{D}}\} = -\imath\Theta_{C\bar{a}}\delta^{D\bar{A}}, \tag{9}$$

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath\left(R_{A\bar{D}}\delta^{B\bar{C}} - R_{C\bar{B}}\delta^{D\bar{A}}\right), \tag{10}$$

where

$$h_{a\bar{b}} = \bar{v}^a v^b, \qquad \Theta_{A\bar{a}} = \bar{\eta}^A v^a, \qquad R_{A\bar{B}} = \imath\bar{\eta}^A\eta^B. \tag{11}$$

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The $u(1)$ generator defining the center of $u(1.N|M)$ is given by the expression

$$J = \gamma_{a\bar{b}} v^a \bar{v}^b + \imath\eta^A\bar{\eta}^A : \quad \{J, h_{a\bar{b}}\} = \{J, \Theta_{A\bar{a}}\} = \{J, R_{A\bar{B}}\} = 0. \tag{12}$$

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

Hence, reducing the system by the action of generator J we will get the "non-compact" projective super-space $\widetilde{\mathbb{CP}}^{N|M}$ (i.e. the supergeneralization of non-compact projective space $\widetilde{\mathbb{CP}}^{N}$), which is $(2N|2M)$-(real)dimensional space.

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For performing the reduction by the action of generator J we have to choose, at first, the $2N$ real (N complex) bosonic and $2M$ real (N complex) fermionic functions commuting with J. Then, we have to calculate their Poisson brackets and restrict the them to the level surface

$$J = g. \tag{13}$$

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As a result we will get the Poisson brackets on the reduced $(2N|2M)$-(real) dimensional space, with that $u(1)$-invariant functions playing the role of the latter's coordinates.

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

The required functions could be easily found,

$$w = \frac{v^N}{v^0}, \quad z^\alpha = \frac{v^\alpha}{v^0}, \quad \theta^A = \frac{\eta^A}{v^0} \ : \quad \{w, J\} = \{z^a, J\} = \{\theta^A, J\} = 0. \tag{14}$$

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Notation:

$$A := \left. \frac{1}{v^0 \bar{v}^0} \right|_{J=g} = \frac{1}{g} \left(\imath(w - \bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma \bar{z}^\gamma + \imath \sum_{C=1}^{M} \theta^C \bar{\theta}^C \right), \tag{15}$$

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we get the reduced Poisson brackets

$$\{w, \bar{w}\} = -A(w - \bar{w}), \quad \{z^\alpha, \bar{z}^\beta\} = \imath A \delta^{\alpha\bar{\beta}}, \quad \{\theta^A, \bar{\theta}^B\} = A\delta^{A\bar{B}}, \tag{16}$$

$$\{w, \bar{z}^\alpha\} = A\bar{z}^\alpha, \quad \{w, \bar{\theta}^A\} = A\bar{\theta}^A. \tag{17}$$

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

In what follows we will call this space "noncompact projective superspace $\widetilde{\mathbb{CP}}^{N|M}$". The isometry algebra of this space is $su(N,1|M)$, which can be easily obtained by the restriction of $u(N.1|M) = u(1) \times su(N.1|M)$ algebra to the level surface $J = g$.

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It is defined by the following Killing potentials

$$H := v^N \bar{v}^N|_{J=g} = \frac{w\bar{w}}{A}, \quad K := v^0 \bar{v}^0|_{J=g} = \frac{1}{A}, \quad D := (v^N \bar{v}^0 + v^0 \bar{v}^N)|_{J=g} = \frac{w+\bar{w}}{A},$$

$$H_\alpha := \bar{v}^\alpha v^N|_{J=g} = \frac{\bar{z}^\alpha w}{A}, \quad K_\alpha := \bar{v}^\alpha v^0|_{J=g} = \frac{\bar{z}^\alpha}{A}, \quad h_{\alpha\bar{\beta}} := \bar{v}^\alpha v^\beta|_{J=g} = \frac{\bar{z}^\alpha z^\beta}{A},$$

$$Q_A := \bar{\eta}^A v^N|_{J=g} = \frac{\bar{\theta}^A w}{A}, \quad S_A := \bar{\eta}^A v^0|_{J=g} = \frac{\bar{\theta}^A}{A}, \quad \Theta_{A\bar{\alpha}} := \bar{\eta}^A v^\alpha|_{J=g} = \frac{\bar{\theta}^A z^\alpha}{A},$$

$$R_{A\bar{B}} := \imath\bar{\eta}^A \eta^B|_{J=g} = \imath\frac{\bar{\theta}^A \theta^B}{A}.$$

$\widetilde{\mathbb{CP}}^{N|M}$ by super-Hamiltonian reduction

- Constructed super-Kähler structure can be viewed as a higher dimensional analog of the Klein model of Lobachevsky space, where the latter is parameterized by the lower half-plane.

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- One can choose, instead of non-diagonal matrix , the diagonal one, $\gamma_{a\bar{b}} = diag(1, -1, \ldots, -1)$. In that case the reduced Kähler structure will have the Fubini-Study-like form.

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- Presented choice $\gamma_{a\bar{b}}$ is motivated by the by its convenience for the analizing superconformal mechanics. In that case the generators H, D, K define conformal subalgebra $su(1.1)$ and are separated from the rest $su(1, N)$ generators.

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- Thus they can be interpreted as the Hamiltonian of conformal mechanics, the generator of conformal boosts and the generator of dilatation.

$su(1, N|M)$ Superconformal Algebra

"Bosonic" sector: $su(1, N) \times u(M)$

$su(1, N|M)$ Superconformal Algebra

"Bosonic" sector: $su(1, N) \times u(M)$

Explicitly, the $su(1.N)$ algebra is given by the relations

$$\{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K,$$
$$\{H, K_\alpha\} = -H_\alpha, \quad \{H, H_\alpha\} = \{H, h_{\alpha\bar\beta}\} = 0,$$
$$\{K, H_\alpha\} = K_\alpha, \quad \{K, K_\alpha\} = \{K, h_{\alpha\bar\beta}\} = 0,$$
$$\{D, K_\alpha\} = -K_\alpha, \quad \{D, H_\alpha\} = H_\alpha, \quad \{D, h_{\alpha\bar\beta}\} = 0,$$
$$\{K_\alpha, K_\beta\} = \{H_\alpha, H_\beta\} = \{K_\alpha, H_\beta\} = 0,$$
$$\{K_\alpha, \overline{K}_\beta\} = -\imath K\delta_{\alpha\bar\beta}, \quad \{H_\alpha, \overline{H}_\beta\} = -\imath H\delta_{\alpha\bar\beta},$$
$$\{K_\alpha, h_{\beta\bar\gamma}\} = -\imath K_\beta\delta_{\alpha\bar\gamma}, \quad \{H_\alpha, h_{\beta\bar\gamma}\} = -\imath H_\beta\delta_{\alpha\bar\gamma},$$
$$\{h_{\alpha\bar\beta}, h_{\gamma\bar\delta}\} = \imath(h_{\alpha\bar\delta}\delta_{\gamma\bar\beta} - h_{\gamma\bar\beta}\delta_{\alpha\bar\delta}), \quad \{K_\alpha, \overline{H}_\beta\} = h_{\alpha\bar\beta} + \frac{1}{2}\left(I - \imath D\right)\delta_{\alpha\bar\beta},$$

where

$$I := g + \sum_{\gamma=1}^{N-1} h_{\gamma\bar{\gamma}} + \sum_{C=1}^{M} R_{C\bar{C}} \tag{18}$$

$su(1, N|M)$ Superconformal Algebra

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The R-symmetry generators form $u(M)$ algebra and commutes with all generators of $su(1, N)$:

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath(R_{A\bar{D}}\delta_{C\bar{B}} - R_{C\bar{B}}\delta_{A\bar{D}}), \quad \{R_{A\bar{B}}, (H; K; D; K_\alpha; H_\alpha; h_{\alpha\bar{\beta}})\} = 0.$$

H, D, K form conformal algebra $su(1, 1)$,

$su(1, N|M)$ Superconformal Algebra

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$su(1, N|M)$ Superconformal Algebra

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H, D, K form conformal algebra $su(1, 1)$,the generators $h_{\alpha\bar{\beta}}$ form the algebra $u(N-1)$, and all together - the $su(1, 1) \times u(N-1)$ algebra. Notice, I in defines the Casimir of conformal algebra $su(1, 1)$:

$$\mathcal{I} := \frac{1}{2}I^2 = \frac{1}{2}D^2 - 2HK. \tag{19}$$

$H \quad \to$ Hamiltonian, $\quad \Rightarrow H_\alpha, h_{\alpha\bar{\beta}}, R_{A\bar{B}}$ constants of motion.
$K \quad \to$ Hamiltonian, $\quad \Rightarrow K_\alpha, h_{\alpha\bar{\beta}}, R_{A\bar{B}}$ constants of motion.

$su(1, N|M)$ Superconformal Algebra

"Fermionic" sector

The Poisson brackets between fermionic generators are as follows

$$\{S_A, \overline{S}_B\} = K\delta_{A\bar{B}}, \quad \{Q_A, \overline{Q}_B\} = H\delta_{A\bar{B}},$$
$$\{S_A, \overline{Q}_B\} = -\imath R_{A\bar{B}} + \frac{\imath}{2}(I - \imath D)\,\delta_{A\bar{B}}, \quad \{\Theta_{A\bar{\alpha}}, \overline{\Theta}_{B\bar{\beta}}\} = R_{A\bar{B}}\delta_{\beta\bar{\alpha}} + h_{\beta\bar{\alpha}}\delta_{A\bar{B}},$$
$$\{S_A, \overline{\Theta}_{B\bar{\alpha}}\} = K_\alpha\delta_{A\bar{B}}, \quad \{Q_A, \overline{\Theta}_{B\bar{\alpha}}\} = H_\alpha\delta_{A\bar{B}},$$

$$\{S_A, S_B\} = \{Q_A, Q_B\} = \{\Theta_{A\bar{\alpha}}, \Theta_{B\bar{\beta}}\} = \{S_A, Q_B\} = \{S_A, \Theta_{B\bar{\alpha}}\} = \{Q_A, \Theta_{B\bar{\alpha}}\} = 0.$$

$su(1,N|M)$ Superconformal Algebra

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$$\{S_A, S_B\} = \{Q_A, Q_B\} = \{\Theta_{A\bar{\alpha}}, \Theta_{B\bar{\beta}}\} = \{S_A, Q_B\} = \{S_A, \Theta_{B\bar{\alpha}}\} = \{Q_A, \Theta_{B\bar{\alpha}}\} = 0.$$

Hence, the functions Q_A play the role of supercharges for the Hamiltonian H, and the functions S_A define the supercharges of the Hamiltonian K playng the role of generator of conformal boosts.

"Mixed" sector The mixed sector is given by the relations

$$
\begin{aligned}
&\{H,Q_A\}=\{H,\Theta_{A\bar{\alpha}}\}=0,\quad \{H,S_A\}=-Q_A,\\
&\{K,S_A\}=\{K,\Theta_{A\bar{\alpha}}\}=0,\quad \{K,Q_A\}=S_A,\\
&\{D,S_A\}=-S_A,\quad \{D,Q_A\}=Q_A,\quad \{D,\Theta_{A\bar{\alpha}}\}=0\\
&\{Q_A,\overline{K}_\alpha\}=-\Theta_{A\bar{\alpha}},\quad \{Q_A,H_\alpha\}=\{Q_A,\bar{H}_\alpha\}=\{Q_A,\bar{K}_\alpha\}=\{Q_A,h_{\alpha\bar{\beta}}\}=0,\\
&\{S_A,\overline{H}_\alpha\}=\Theta_{A\bar{\alpha}},\quad \{S_A,K_\alpha\}=\{S_A,\bar{K}_\alpha\}=\{S_A,H_\alpha\}=\{S_A,h_{\alpha\bar{\beta}}\}=0,\\
&\{\Theta_{A\bar{\alpha}},K_\beta\}=\imath S_A\delta_{\beta\bar{\alpha}},\quad \{\Theta_{A\bar{\alpha}},H_\beta\}=\imath Q_A\delta_{\beta\bar{\alpha}},\\
&\{\Theta_{A\bar{\alpha}},\bar{H}_\alpha\}=\{\Theta_{A\bar{\alpha}},\bar{K}_\alpha\}=0,\quad \{\Theta_{A\bar{\alpha}},h_{\beta\bar{\gamma}}\}=\imath\Theta_{A\bar{\gamma}}\delta_{\beta\bar{\alpha}},\\
&\{S_A,R_{B\bar{C}}\}=-\imath S_B\delta_{A\bar{C}},\quad \{Q_A,R_{B\bar{C}}\}=-\imath Q_B\delta_{A\bar{C}},\\
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\end{aligned}
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$su(1, N|M)$ Superconformal Algebra

Looking at all the Poisson bracket relations together we conclude that

- The bosonic functions H_α, $h_{\alpha\bar{\beta}}$, and the fermionic functions Q_A, $\Theta_{A\bar{\alpha}}$ commute with the Hamitonian H and thus, provide it by the superintegrability property

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- The triples $(H, H_\alpha, Q_A,)$ and $(K, K_\alpha, S_A,)$ transform into each other under the discrete transformation

$$(w, z^\alpha, \theta^A) \to (-\frac{1}{w}, \frac{z^\alpha}{w}, \frac{\theta^A}{w}) \quad \Rightarrow D \to -D, \quad \left\{ \begin{array}{ccc} (H, H_\alpha, Q_A,) & \to & (K, -K_\alpha, -S_A), \\ (K, K_\alpha, S_A) & \to & (H, H_\alpha, Q_A,) \end{array} \right. .$$

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- The functions $h_{\alpha\bar{\beta}}, \Theta_{A\bar{\alpha}}$ are invariant under this discrete transformation. Moreover, they appear to be constants of motion both for H and K. Hence, they remain to be constants of motion for any Hamiltonian being the functions of H, K. In particular, adding to the Hamiltonian H the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems with dynamical superconformal symmetry .

Canonical Coordinates

We define canonical coordinates as follows

$$w = \frac{p_r}{r} - \imath \frac{I}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r} \mathrm{e}^{\imath\varphi_\alpha}, \quad \theta^A = \frac{\sqrt{2}}{r}\chi^A,$$

where

$$\{r, p_r\} = 1, \quad \{\varphi_\beta, \pi_\alpha\} = \delta_{\alpha\beta}, \quad \{\chi^A, \bar{\chi}^B\} = \delta^{A\bar{B}}, \qquad \pi_a \geq 0, \quad \varphi^a \in [0, 2\pi), \quad r > 0.$$

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They expresses via initial ones as follows

$$p_r = \frac{w + \bar{w}}{2}\sqrt{\frac{2}{A}}, \quad r = \sqrt{\frac{2}{A}}, \quad \pi_\alpha = \frac{z^\alpha \bar{z}^\alpha}{A}, \quad \varphi_\alpha = \arg(z^\alpha), \quad \chi^A = -\frac{\theta^A}{\sqrt{A}},$$

where

$$I = g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A\,, \qquad A := \frac{\imath(w - \bar{w}) - z^\gamma\bar{z}^\gamma + \imath\theta^C\bar{\theta}^C}{g} = \frac{2}{r^2}.$$

In these canonical coordinates the isometry generators read

$$H = \frac{p_r^2}{2} + \frac{I^2}{2r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r,$$

$$H_\alpha = \sqrt{\frac{\pi_\alpha}{2}} \mathrm{e}^{-\imath\varphi_\alpha} \left(p_r - \imath \frac{I}{r} \right), \quad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}} \mathrm{e}^{-\imath\varphi_\alpha}, \quad h_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha \pi_\beta} \mathrm{e}^{-\imath(\varphi_\alpha - \varphi_\beta)},$$

$$Q_A = \frac{\bar{\chi}^A}{\sqrt{2}} \left(p_r - \imath \frac{\sqrt{2\mathcal{I}}}{r} \right), \quad S_A = \frac{\bar{\chi}^A}{\sqrt{2}} r, \quad \Theta_{A\bar{\alpha}} = \bar{\chi}^A \sqrt{\pi_\alpha} \mathrm{e}^{\imath\varphi_\alpha}, \quad R_{A\bar{B}} = \imath \bar{\chi}^A \chi^B.$$

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Interpreting r as a radial coordinate, and p_r as radial momentum, we get the superconformal mechanics with angular Hamiltonian given by

$$\mathcal{I} := \frac{I_0 + (\bar{\chi}\chi)}{2}, \qquad \text{with} \qquad I_0 := g + \sum_{\alpha=1}^{N-1} \pi_\alpha, \quad (\bar{\chi}\chi) := \sum_{A=1}^{M} \imath \bar{\chi}^A \chi^A \,.$$

Fermionic part of superconformal Hamiltonian is in its angular part.

Canonical Coordinates

The explicit dependence of Hamiltonian H and of its supercharges Q_A and on fermions is as follows

$$H = H_0 + \frac{I_0(\bar\chi\chi)}{r^2} + \frac{(\bar\chi\chi)^2}{2r^2}, \qquad Q_A = -\frac{\bar\chi^A}{\sqrt{2}}\left(p_r - \imath\frac{I_0}{r} - \imath\frac{(\bar\chi\chi)}{r}\right),$$

while the dependence of bosonic integrals H_α on fermions is given by the expression

$$H_\alpha = H_\alpha^0 - \frac{K_\alpha(\bar\chi\chi)}{2K},$$

where

$$H_0 := \frac{p_r^2}{2} + \frac{I_0^2}{2r^2}, \quad H_\alpha^0 = \sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}\left(p_r - \imath\frac{I_0}{r}\right) \ : \quad \{H_\alpha^0, H^0\} = 0.$$

So, proposed superconformal Hamiltonian H inherits all symmetries of initial Hamiltonian H_0 (given by $H_\alpha^0, h_{\alpha\bar\beta}$).

Oscillator-like Systems

We define the supersymmetric oscillator-like system with the phase space $\widetilde{\mathbb{CP}}^{N|M}$ by the Hamiltonian

$$H_{osc} = H + \omega^2 K, \tag{20}$$

In canonical coordinates it reads

$$H_{osc} = \frac{p_r^2}{2} + \frac{(g + \sum_{\alpha=1}^{N-1} \pi_\alpha + \sum_{A=1}^{M} \imath\bar{\chi}^A \chi^A)^2}{r^2} + \frac{\omega^2 r^2}{2}. \tag{21}$$

This system possesses the $u(N)$ symmetry given by the generators $h_{\alpha\bar{\beta}}$ (among them $N-1$ constants of motion π_α are functionally independent),

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$$M_{\alpha\beta} = (H_\alpha + \imath\omega K_\alpha)(H_\beta - \imath\omega K_\beta) = \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2) \ : \quad \{H_{osc}, M_{\alpha\beta}\} = 0,$$

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These generators and the $su(N)$ generators $h_{\alpha\bar{\beta}}$ form the following symmetry algebra

$$\{h_{\alpha\bar{\beta}}, M_{\gamma\delta}\} = \imath \left(M_{\alpha\delta}\delta_{\gamma\bar{\beta}} + M_{\gamma\alpha}\delta_{\delta\bar{\beta}} \right), \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0,$$

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here summation over repeated indices is not assumed.

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here summation over repeated indices is not assumed. Besides, this system has a fermionic constants of motion $\Theta_{A\bar{\alpha}}$.

Hence, it is superintegrable system in the sense of super-Liuville theorem, i.e. it has $2N-1$ bosonic and $2M$ fermionic, functionally independent, constants of motion.

Let us show, that for the even $M = 2k$ this system possess the deformed $\mathcal{N} = 2k$ Poincaré supersymmetry, in the sense of paper E. Ivanov and S. Sidorov, *Deformed Supersymmetric Mechanics*, Class. Quant. Grav. **31** (2014) 075013 [arXiv:1307.7690 [hep-th]].
For this purpose we choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \omega C_{AB}\bar{S}_B, \tag{22}$$

with the constant matrix C_{AB} obeying the conditions

$$C_{AB} + C_{BA} = 0, \qquad C_{AB}\overline{C}_{BD} = -\delta_{A\bar{D}} \tag{23}$$

For sure, the condition (23) assumes that M is an even number, $M = 2k$.

Oscillator-like Systems

Calculating Poisson brackets of the functions (22) we get

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{osc}\delta_{AB}, \qquad \{\mathcal{Q}_A, \mathcal{Q}_B\} = -\imath\omega\mathcal{G}_{AB}, \qquad \{\bar{\mathcal{Q}}_A, \bar{\mathcal{Q}}_B\} = \imath\omega\bar{\mathcal{G}}_{AB},$$

where

$$\mathcal{G}_{AB} := C_{AC}R_{B\bar{C}} + C_{BC}R_{A\bar{C}}, \quad \mathcal{G}_{\bar{A}\bar{B}} := \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}R_{C\bar{B}} + \bar{C}_{BC}R_{C\bar{A}}, \quad \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}\bar{C}_{DB}\mathcal{G}_{DC}.$$

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Then we get that the algebra of generators $\mathcal{Q}_A$, $\mathcal{H}_{osc}$, $\mathcal{G}_{AB}$ is closed indeed:

$$\{\mathcal{Q}_A, H_{osc}\} = \omega C_{AB}\mathcal{Q}_B, \qquad \{\mathcal{G}_{AB}, H_{osc}\} = 0,$$
$$\{\mathcal{Q}_A, \mathcal{G}_{BC}\} = \imath(C_{AB}\mathcal{Q}_C + C_{AC}\mathcal{Q}_B),$$
$$\{\mathcal{Q}_A, \bar{\mathcal{G}}_{BC}\} = -\imath(\bar{C}_{BD}\mathcal{Q}_D\delta_{A\bar{C}} + \bar{C}_{CD}\mathcal{Q}_D\delta_{A\bar{B}}).$$

Hence, for the $M = 2k$ the above oscillator-like system possesses deformed $\mathcal{N} = 4k$ supersymmetry.

Let us present other deformed $\mathcal{N} = 2M$ Poincaré supersymmetryc systems whose bosonic part is different, but nevertheless, has the oscillator potential.

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These supercharges generates the $su(1|M)$ superalgebra, and thus generalizes the systems considered in paper by Ivanov and Sidorov to arbitrary M,

$$\{\widetilde{\mathcal{Q}}_A, \bar{\widetilde{\mathcal{Q}}}_B\} = \mathcal{H}_{osc}\delta_{AB} - \omega\mathcal{R}_B^A, \quad \{\mathcal{R}_A^{\ B}, \mathcal{R}_C^{\ D}\} = \imath(\mathcal{R}_A^{\ D}\delta_C^B - \mathcal{R}_C^{\ B}\delta_A^D)$$
$$\{\widetilde{\mathcal{Q}}_A, \mathcal{R}_B^C\} = \imath\left(\tfrac{1}{M}\widetilde{\mathcal{Q}}_A\delta_{B\bar{C}} + \widetilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right), \quad \{\widetilde{\mathcal{Q}}_A, \mathcal{H}_{osc}\} = \imath\omega\tfrac{2M-1}{M}\widetilde{\mathcal{Q}}_A,$$

where $\mathcal{H}_{osc} := H_{osc} - \omega(I + \frac{1}{M}\sum_C R_{C\bar{C}}), \quad \mathcal{R}_A^{\ B} := R_{A\bar{B}} - \frac{1}{M}\delta_A^B\sum_C R_{C\bar{C}}$.
Hence, the Hamiltonian get the additional bosonic term proportional to the casimir of conformal group.

Coulomb-like Systems

We define the supersymmetric Coulomb-like system with the phase space $\widetilde{\mathbb{CP}}^{N|M}$ by the Hamiltonian

$$H_{Coul} = H + \frac{\gamma}{\sqrt{2K}}. \quad (25)$$

The bosonic constants of motion of this system are given by the $u(N-1)$ symmetry generators $h_{\alpha\beta}$, and by the $N-1$ additional constants of motion

$$R_\alpha = H_\alpha + \imath\gamma \frac{K_\alpha}{I\sqrt{2K}} \quad : \quad \{H_{Coul}, R_\alpha\} = \{H_{Coul}, h_{\alpha\bar{\beta}}\} = 0.$$

These generators form the following algebra

$$\{R_\alpha, \bar{R}_{\bar{\beta}}\} = -\imath\delta_{\alpha\bar{\beta}}\left(H_{Coul} - \frac{\imath\gamma^2}{2I^2}\right) + \frac{\imath\gamma^2 h_{\alpha\bar{\beta}}}{2I^3}, \quad \{h_{\alpha\bar{\beta}}, R_\gamma\} = \imath\delta_{\gamma\bar{\beta}} R_\alpha, \quad \{R_\alpha, R_\beta\} = 0.$$

Coulomb-like Systems

Besides, proposed system has $2M$ fermionic constants of motion given by $\Theta_{A\bar{\alpha}}$, and $u(M)$ R-symmetry charges given by $R_{A\bar{B}}$.

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However, it is not a case.
So, proposed superextensions of Coulomb-like systems, being well-defined from the viewpoint of superintegrability, do not possess neither $\mathcal{N}=2M$ supersymmetry, nor its deformation. The $su(1,N|M)$ superalgebra plays the role of dynamical algebra of that systems.

Conclusion

- We have suggested to construct the $su(1, N|M)$-superconformal mechanics formulating them on phase superspace given by the non-compact analog of complex projective superspace $\mathbb{CP}^{N|M}$. The $su(1, N|M)$ symmetry generators were defined there as a Killing potentials of $\mathbb{CP}^{N|M}$.

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- Then we transited to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics.
- We also proposed the superintegrable oscillator- and Coulomb- like systems with a $su(1,N|M)$ dynamical superalgebra, and found that oscillator-like systems admits deformed $\mathcal{N}=2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

The End

Thank You!